Shortest Paths 3

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Some contents of this presentation are adapted from year 2005 course notes for 620-362 Applied Operations Research, Department of Mathematics and Statistics, The University of Melbourne (compiled by Prof Natashia Boland and Dr Renata Sotirov)

Outline

Equipment Replacement Strategy

It costs $19,000 to buy a new car.
Each year a car becomes more expensive to run.

Age Of Car	Annual Costs	Trade-In Value
0	$1,000	$12,000
1	$2,000	$10,000
2	$5,000	$6,000
3	$9,000	$5,000
4	$12,000	$3,000

We aim to determine the best strategy for the next 5 years.

Equipment Replacement Strategy

New car prices

Start Of Year	Price
1	$19,000
2	$18,500
3	$18,000
4	$18,000
5	$17,000

Our Model

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The cost of arc (i, j) is the cost of buying a car in year i plus the maintenance cost for $j - i$ years less the resale value of a $(j - i)$ year old car.

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From a complexity point of view, DP and OP are equivalent.

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while $U \neq L$ **do**
  $M = \left\lfloor \frac{U+L}{2} \right\rfloor$
  if $DP(M) =$ Yes **then**
    $U = M$
  else
    $L = M + 1$
  end if
end while

Outline

System Of Difference Equations

We will refer to a problem as a system of difference equations if it has the following form:

Variables are $x_1, x_2, \ldots, x_n$.
Constraints are of the form $x_{j_k} - x_{i_k} \leq b_k \quad \forall k = 1 \ldots m$
Example:

$$
\begin{aligned}
x_1 - x_2 &\leq 9 \\
x_2 - x_3 &\leq -4 \\
x_3 - x_4 &\leq -2 \\
x_4 - x_5 &\leq -3 \\
x_5 - x_1 &\leq 2 \\
x_2 - x_5 &\leq 3 \\
x_3 - x_1 &\leq 4
\end{aligned}
$$

Constraint Graph

In the Constraint Graph, each variable is represented by a node. Each constraint $x_j - x_i \leq b$ is represented by an arc from node i to node j with weight b.

Proposition

A system of difference constraints has a feasible solution if and only if its constraint graph has no negative length cycles.

Proof.

See lecture. □

We also define the Augmented Constraint Graph which has an additional dummy node 0 and an arc $(0, i)$ for each node i with cost 0.

Example

$$
\begin{aligned}
x_1 - x_2 &\leq 3 \\
x_2 - x_3 &\leq -4 \\
x_3 - x_4 &\leq -2 \\
x_4 - x_5 &\leq -3 \\
x_5 - x_1 &\leq 2 \\
x_2 - x_5 &\leq 3 \\
x_3 - x_1 &\leq 4
\end{aligned}
$$

If the constraint graph has negative cycles, then the system is infeasible.

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Otherwise, we set the cost of each variable to the cost of the shortest path from node 0 to its node. This gives a feasible solution.

A Work Scheduling Problem

A business requires r_i staff to work on each day $i = 1 \ldots 7$ of the week.

The staff prefer to work 5 days on and then 2 days off.

We aim to minimise the total number of staff required to cover each day.

Work Scheduling Model

Let x_i be the number of employees that start working on day i.

$$
\begin{aligned}
\min \quad & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \\
& x_1 + x_2 + x_3 + x_4 + x_5 \geq r_1 \\
& x_2 + x_3 + x_4 + x_5 + x_6 \geq r_2 \\
& x_3 + x_4 + x_5 + x_6 + x_7 \geq r_3 \\
& x_4 + x_5 + x_6 + x_7 + x_1 \geq r_4 \\
& x_5 + x_6 + x_7 + x_1 + x_2 \geq r_5 \\
& x_6 + x_7 + x_1 + x_2 + x_3 \geq r_6 \\
& x_7 + x_1 + x_2 + x_3 + x_4 \geq r_7 \\
& x_i \text{ integer} \quad \forall i = 1 \ldots 7
\end{aligned}
$$

New Decision Variables

Let y_i be the number of people starting work on day $1 \ldots i$.

$$
\begin{aligned}
y_1 &= x_1 \\
y_2 &= x_1 + x_2 \\
y_3 &= x_1 + x_2 + x_3 \\
y_4 &= x_1 + x_2 + x_3 + x_4 \\
y_5 &= x_1 + x_2 + x_3 + x_4 + x_5 \\
y_6 &= x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \\
y_7 &= x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7
\end{aligned}
$$

Modelling With Our New Variables

$$
\begin{aligned}
\min \quad & y_7 \\
& y_5 \geq r_1 \\
& y_6 - y_1 \geq r_2 \\
& y_7 - y_2 \geq r_3 \\
& y_7 - y_3 + y_1 \geq r_4 \\
& y_7 - y_4 + y_2 \geq r_5 \\
& y_7 - y_5 + y_3 \geq r_6 \\
& y_7 - y_6 + y_4 \geq r_7 \\
& y_7 \geq y_6 \geq y_5 \geq y_4 \geq y_3 \geq y_2 \geq y_1 \geq 0
\end{aligned}
$$

Decision Model

We will use the bisection method to solve this problem, requiring $y_7 \leq M$.
We introduce a dummy variable y_0 and set $y_7 - y_0 = M$.
If there is any solution to our eventual system of difference equations, then there is a solution with $y_0 = 0$, so we assume $y_7 = M$.

Decision Model

$$y_7 - y_5 \leq M - r_1$$
$$y_1 - y_6 \leq -r_2$$
$$y_2 - y_7 \leq -r_3$$
$$y_3 - y_1 \leq M - r_4$$
$$y_4 - y_2 \leq M - r_5$$
$$y_5 - y_3 \leq M - r_6$$
$$y_6 - y_4 \leq M - r_7$$

Decision Model Continued

$$y_6 - y_7 \leq 0$$
$$y_5 - y_6 \leq 0$$
$$y_4 - y_5 \leq 0$$
$$y_3 - y_4 \leq 0$$
$$y_2 - y_3 \leq 0$$
$$y_1 - y_2 \leq 0$$
$$y_0 - y_1 \leq 0$$