

Network Flow Problems 1

October 17, 2008

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Some contents of this presentation are adapted from year 2005 course notes for 620-362 Applied Operations Research, Department of Mathematics and Statistics, The University of Melbourne (compiled by Prof Natasha Boland and Dr Renata Sotirov)

Outline

- 1 Definitions
- 2 Modelling with Network Flow
- 3 Multi-Commodity Flow

Definitions

Each arc a in a network is considered to carry a flow x_a . A negative flow is considered to be flow in the opposite direction to the arc.

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The **inflow** at a node j :

$$\begin{aligned} &= \text{total flow in to node } j - \text{total flow out of node } j \\ &= \sum_{i|(i,j) \in A} x_{i,j} - \sum_{k|(j,k) \in A} x_{j,k} \end{aligned}$$

Feasible Flow

Given a network (N, A) , a vector of required inflows b at each node, upper and lower bounds u and l on the allowable flow on each arc, a flow is defined to be feasible if

$$\sum_{i|(i,j) \in A} x_{i,j} - \sum_{k|(j,k) \in A} x_{j,k} = b_j \quad \forall j \in N \quad (\text{Inflow})$$

$$l_{i,j} \leq x_{i,j} \leq u_{i,j} \quad \forall (i,j) \in A \quad (\text{Capacity})$$

Feasible Flow

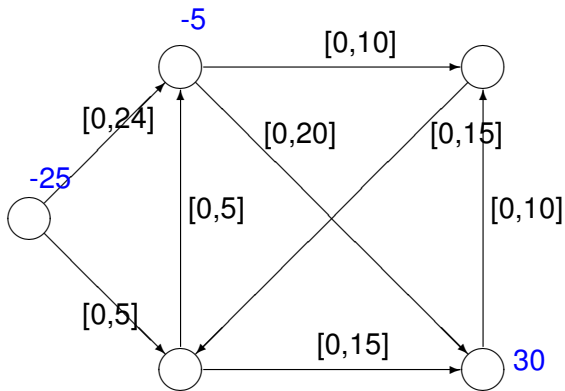
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What does this mean about the total required inflow?

Example



Optimisation Problem

We allow each arc (i, j) to have a cost $c_{i,j}$ per unit of flow.

This gives the following optimisation problem:

$$\min_x \sum_{(i,j) \in A} c_{i,j} x_{i,j} \quad (1)$$

$$\text{s.t.} \quad \sum_{i|(i,j) \in A} x_{i,j} - \sum_{k|(j,k) \in A} x_{j,k} = b_j \quad \forall j \in N \quad (2)$$

$$l_{i,j} \leq x_{i,j} \leq u_{i,j} \quad \forall (i, j) \in A \quad (3)$$

$$x_{i,j} \text{ integer} \quad \forall (i, j) \in A \quad (4)$$

The Good News

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Network Flow has the **integrality property**. Which means that if b , l and u are integer, an optimal solution to its LP relaxation will be integer.

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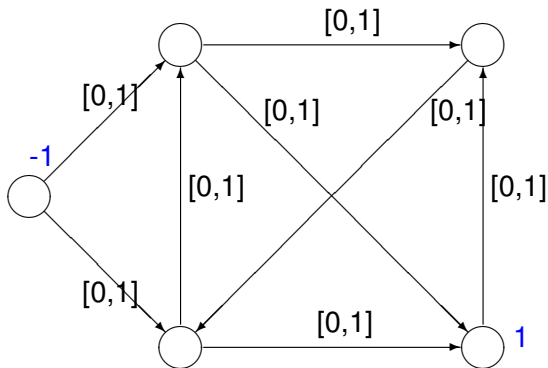
We have methods which are even faster than the simplex method for solving network flow problems.

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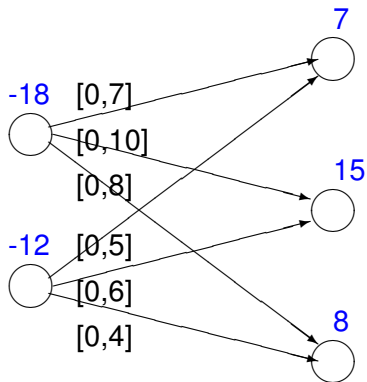
The shortest path problem.

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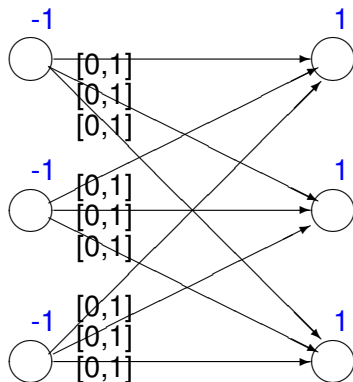
Transportation Problem

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Assignment Problem

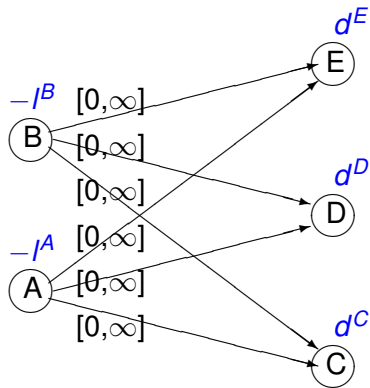
Assignment Problem



Distribution

- Two plants producing goods, A and B .
- Three customers requiring goods, C , D and E .
- Production costs c^A , c^B .
- Production amounts I^A , I^B .
- Transport costs t^{XY} .
- Demand d^C , d^D , d^E .

Distribution

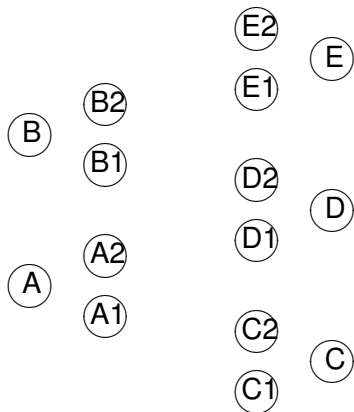


Distribution Extensions

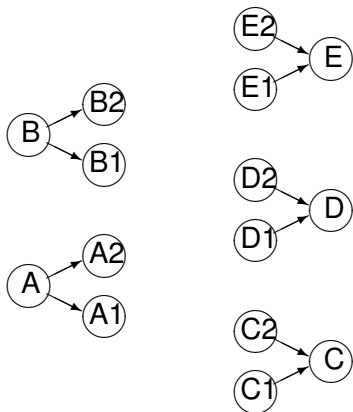
What if the capacities are upper bounds rather than exact values?

What if there are multiple products?

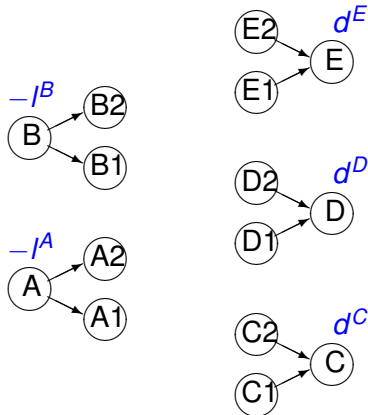
Distribution with Multiple Products



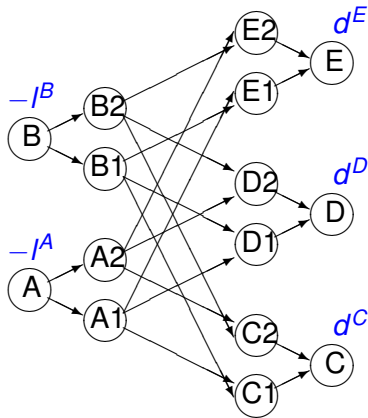
Distribution with Multiple Products



Distribution with Multiple Products



Distribution with Multiple Products



Inventory Problem

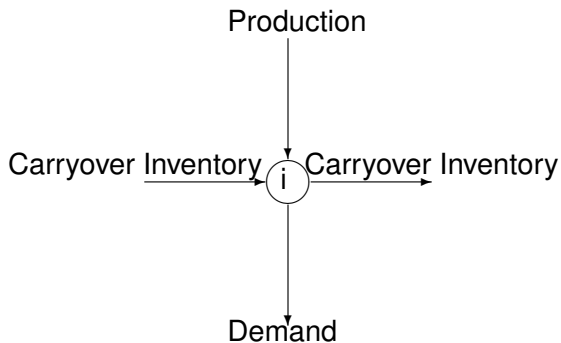
We need to meet set demand for each time period.

We can produce no more than a set limit in each time period.

We incur costs for holding units in inventory and for production.

How can we model this as a network flow problem?

A Single Time Period



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What about if we receive 'economies of scale' benefits for large production?

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The commodities share capacity.

LP Formulation

Standard Network Flow

$$\begin{array}{ll} \min & cx \\ \text{s.t.} & Mx = b \\ & l \leq x \leq u \end{array}$$

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Multicommodity Network Flow

$$\begin{aligned} \min \quad & \sum_{k \in K} c^k x^k \\ \text{s.t.} \quad & Mx^k = b^k \\ & l^k \leq x^k \leq u^k \\ & l^a \leq \sum_{k \in K} x^k \leq u^a \end{aligned}$$

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Multi-commodity flow is a very expressive modelling tool.