

620-362 Applied Operations Research

Two-Stage Stochastic Programming

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Some contents of this presentation are adapted from year 2005 course notes for 620-362 Applied Operations Research, Department of Mathematics and Statistics, The University of Melbourne (compiled by Prof Natasha Boland and Dr Renata Sotirov)

The Farmer's Problem

- Farmer (in Europe) raises wheat, corn and beets.
- 500 acres available for cultivation
- Needs 200 tonnes wheat and 240 tonnes corn for cattle feed
- Excess production can be sold; European Commission has 600 tonne quota on beets and selling price drops over quota
- **How much land should be used for production of each type of crop to maximise farmer's profit?**

Farmer's Data

Yield is for an average year

| | Wheat | Corn | Beets |
|--|-------|------|---|
| Yield (T/acre) | 2.5 | 3 | 20 |
| Planting cost (\$/acre) | 150 | 230 | 260 |
| Selling price (\$/T) | 170 | 150 | 36 if ≤ 6000 T 10 if > 6000 T |

What should the farmer do?

- Calculate profit per acre for each crop:

| | Wheat | Corn | Beets | |
|---------------------------|-----------------------------------|---------------------------------|---------------------------------|---------------------------------|
| Profit \$/acre | $170 \times 2.5 - 150$ $= 275$ | $150 \times 3 - 230$ $= 220$ | $36 \times 20 - 260$ $= 460$ | $10 \times 20 - 260$ $= -60$ |

- Raise as much beet as possible \leq quota
- Need ≥ 80 acres each for wheat and corn
- 340 acres yields 6800 tonnes of beets
- Plant 300 acres beets, 120 acres wheat, 80 acres corn

The Farmer's plan based on an Average Year

| | Wheat | Corn | Beets |
|--------------|-------|------|-------|
| Acres | 120 | 80 | 300 |
| Yield (T) | 300 | 240 | 6000 |
| Sales (T) | 100 | - | 6000 |

Total cost = \$114,400

Total sale = \$233,000

Total profit = \$118,600

A Bad Year Occurs

- All crop yields are 20% below expected
- Assume prices unaffected

| | Wheat | Corn | Beets |
|------------------|--------------|-------------|--------------|
| Acres | 120 | 80 | 300 |
| Yield (T) | 240 | 192 | 4800 |
| Sales (T) | 40 | - | 4800 |

- Cannot meet cattle requirements for corn!
- Need to buy it in: wheat \$238/T, corn \$210/T
- Total profit = \$55,120

What if the farmer had planned for a bad year?

An LP to help the farmer plan:

Decision variables:

x_w = acres planted with wheat

x_c = acres planted with corn

x_b = acres planted with beets

s_w = tonnes of wheat sold

s_c = tonnes of corn sold

s_b^H = tonnes of beets sold at high price

s_b^L = tonnes of beets sold at low price

p_w = tonnes of wheat purchased

p_c = tonnes of corn purchased

What if the farmer had planned for a bad year?

An LP to help the farmer plan:

Objective: Maximise profit

$$\text{profit from wheat} = 170s_w - 238p_w - 150x_w$$

$$\text{profit from corn} = 150s_c - 210 p_c - 230x_c$$

$$\text{profit from beets} = 36s_b^H + 10s_b^L - 260x_b$$

Total profit =

$$170s_w - 238p_w - 150x_w + 150s_c - 210 p_c - 230x_c + 36s_b^H + 10s_b^L - 260x_b$$

What if the farmer had planned for a bad year?

Constraints:

$$\text{total acres planted} = x_w + x_c + x_b \leq 500$$

$$\text{net wheat in hand (tonnes)} = 2x_w + p_w - s_w \geq 200$$

$$\text{net corn in hand (tonnes)} = 2.4x_c + p_c - s_c \geq 240$$

$$\text{tonnes beets sold} = s_b^H + s_b^L \leq 16x_b$$

$$\text{tonnes beets sold at high price} = s_b^H \leq 6000$$

The Farmer's LP for a Bad Year

$$\begin{aligned} \max \quad & 170s_w - 238p_w - 150x_w + 150s_c - 210p_c - 230x_c \\ & + 36s_c^H + 10s_c^L - 260x_b \end{aligned}$$

s.t.

$$x_w + x_c + x_b \leq 500$$

$$2x_w + p_w - s_w \geq 200$$

$$2.4x_c + p_c - s_c \geq 240$$

$$s_b^H + s_b^L \leq 16x_b$$

$$s_b^H \leq 6000$$

$$x_w, x_c, x_b, p_w, p_c, s_w, s_c, s_b^H, s_b^L \geq 0$$

The Farmer's Plan Based on a Bad Year

| | Wheat | Corn | Beets |
|---------------------|-------|------|-------|
| Acres | 100 | 25 | 375 |
| Yield (T) | 200 | 60 | 6000 |
| Sales (T) | - | - | 6000 |
| Purchase (T) | - | 180 | - |

Total profit = \$59,950

Value of perfect forecast = \$59,950 - \$55,120
= \$4,830

Farmer plan for average: a Good Year Occurs

- All crop yields are 20% above expected
- Assume prices unaffected

| | Wheat | Corn | Beets |
|------------------|-------|------|------------------------------------|
| Acres | 120 | 80 | 300 |
| Yield (T) | 360 | 288 | 7200 |
| Sales (T) | 160 | 48 | 6000 at 36 \$/T 1200 at 10 \$/T |

- Using acres for very unprofitable beets!
- Total profit = \$148,000

The Farmer's LP for a Good Year

$$\begin{aligned} \max \quad & 170s_w - 238p_w - 150x_w + 150s_c - 210p_c - 230x_c \\ & + 36s_c^H + 10s_c^L - 260x_b \end{aligned}$$

s.t.

$$x_w + x_c + x_b \leq 500$$

$$3x_w + p_w - s_w \geq 200$$

$$3.6x_c + p_c - s_c \geq 240$$

$$s_b^H + s_b^L \leq 24x_b$$

$$s_b^H \leq 6000$$

$$x_w, x_c, x_b, p_w, p_c, s_w, s_c, s_b^H, s_b^L \geq 0$$

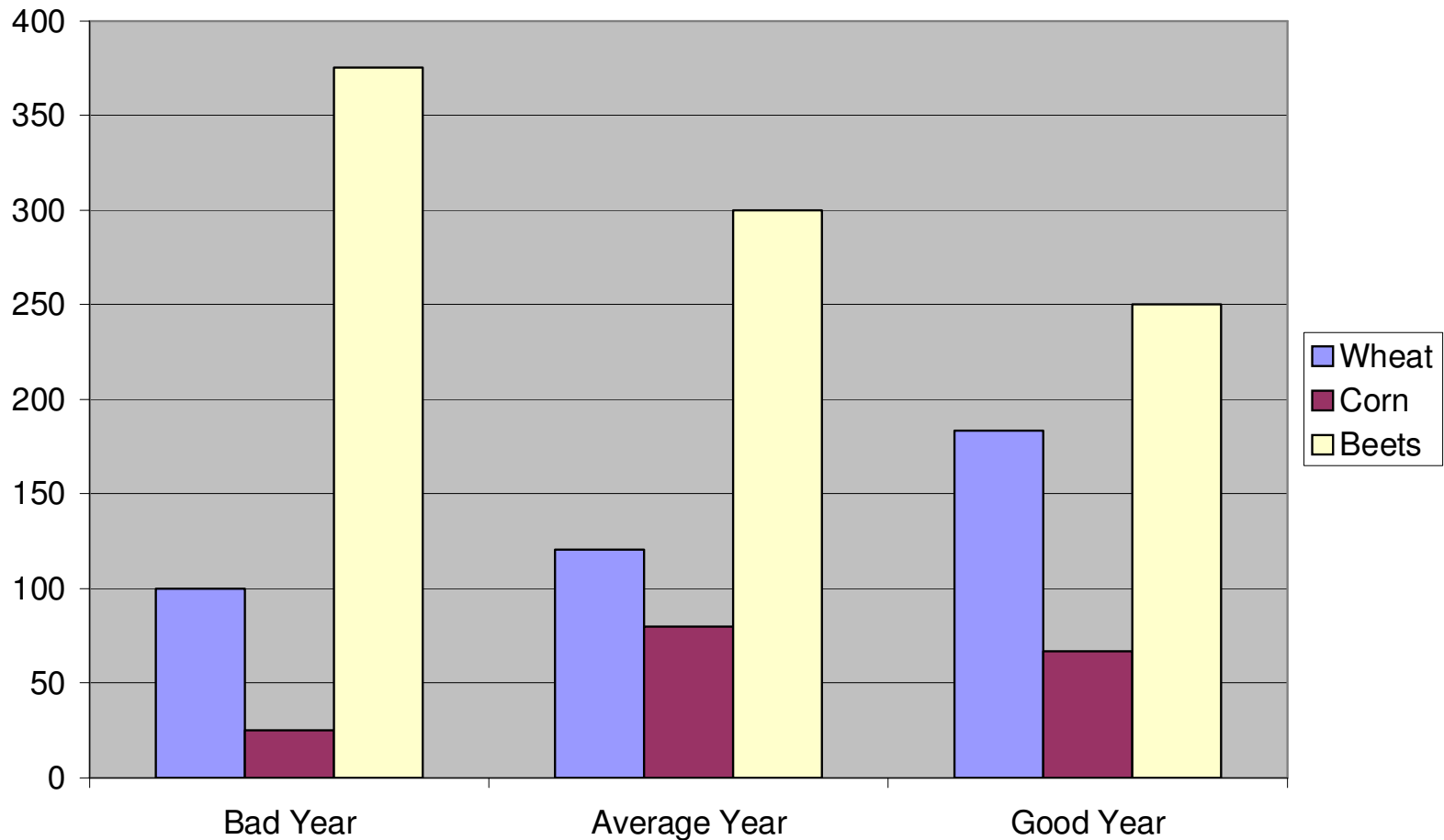
The Farmer's Plan Based on a Bad Year

| | Wheat | Corn | Beets |
|---------------------|--------|-------|-------|
| Acres | 183.33 | 66.67 | 250 |
| Yield (T) | 550 | 240 | 6000 |
| Sales (T) | 350 | - | 6000 |
| Purchase (T) | - | - | - |

Total profit = \$167,667

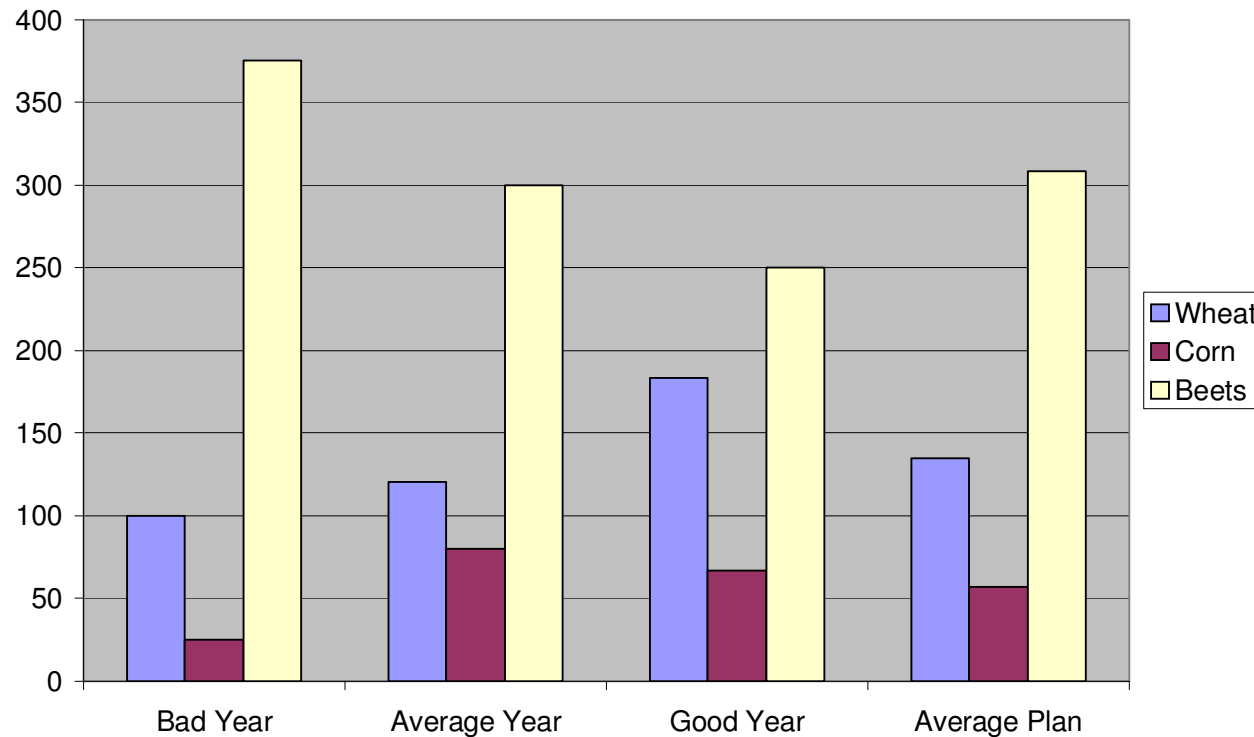
Value of perfect forecast = \$167,667 - \$148,000
= \$19,667

Plan based on each scenario



If all scenarios are equally likely, what should the farmer do?

Take the Average Plan?



Average acres wheat = $(100+120+183.33)/3 = 134.44$

Average acres corn = $(25+80+66.67)/3 = 57.22$

Average acres beets = $(375+300+250)/3 = 308.33$

Expected Profit from the Average Plan

| | | Wheat | Corn | Beets | | Overall |
|---|--------------|---------|-----------|----------|---------|----------|
| | | | | <= 6000T | > 6000T | |
| Yield (T/acre) | | 2.5 | 3 | 20 | | |
| Planting cost (\$/acre) | | 150 | 230 | 260 | | |
| Selling price (\$/T) | | 170 | 150 | 36 | 10 | |
| Additional purchase price (\$/T) | | 238 | 210 | | | |
| Acres | | 134.44 | 57.22 | 308.33 | | |
| Planting cost (\$) | | 20166 | 13160.6 | 80165.8 | | |
| Bad Year | Yield (T) | 268.88 | 137.328 | 4933.28 | | |
| | Sales (T) | 68.88 | | 4933.28 | | |
| | Purchase (T) | | 102.672 | | | |
| | Profit (\$) | -8456.4 | -34721.72 | 97432.28 | | 54254.16 |
| Average Year | Yield (T) | 336.1 | 171.66 | 6166.6 | | |
| | Sales (T) | 136.1 | | 6166.6 | | |
| | Purchase (T) | | 68.34 | | | |
| | Profit (\$) | 2971 | -27512 | 137500.2 | | 112959.2 |
| Good Year | Yield (T) | 403.32 | 205.992 | 7399.92 | | |
| | Sales (T) | 203.32 | | 7399.92 | | |
| | Purchase (T) | | 34.008 | | | |
| | Profit (\$) | 14398.4 | -20302.28 | 149833.4 | | 143929.5 |

Expected Profit

$$= (\$54,254 + \$112,959 + \$143,930)/3$$

$$= \$103,714$$

Expected Profit from Plan for Average Year

| | | Wheat | Corn | Beets | | Overall |
|---|--------------|--------|--------|----------|---------|---------|
| | | | | <= 6000T | > 6000T | |
| Yield (T/acre) | | 2.5 | 3 | 20 | | |
| Planting cost (\$/acre) | | 150 | 230 | 260 | | |
| Selling price (\$/T) | | 170 | 150 | 36 | 10 | |
| Additional purchase price (\$/T) | | 238 | 210 | | | |
| Acres | | 120 | 80 | 300 | | |
| Planting cost (\$) | | 18000 | 18400 | 78000 | | |
| Bad Year | Yield (T) | 240 | 192 | 4800 | | |
| | Sales (T) | 40 | | 4800 | | |
| | Purchase (T) | | 48 | | | |
| | Profit (\$) | -11200 | -28480 | 94800 | | 55120 |
| Average Year | Yield (T) | 300 | 240 | 6000 | | |
| | Sales (T) | 100 | | 6000 | | |
| | Purchase (T) | | | | | |
| | Profit (\$) | -1000 | -18400 | 138000 | | 118600 |
| Good Year | Yield (T) | 360 | 288 | 7200 | | |
| | Sales (T) | 160 | 48 | 7200 | | |
| | Purchase (T) | | | | | |
| | Profit (\$) | 9200 | -11200 | 150000 | | 148000 |

Expected Profit

$$= (\$55,120 + \$118,600 + \$148,000)/3$$

$$= \$107,240$$

Stochastic Optimization

- No set of decisions is perfect for all scenarios
- Farmer can hedge against different scenarios using stochastic optimization
- Some decisions can be delayed until after the scenario eventuates: stage 1 and stage 2 decisions
- Assume the farmer is risk neutral
- How can the farmer make decisions now that will maximise the farm's long-run profit?

Stochastic Modelling

- Decisions now (stage 1 decisions):
 - number of acres to plant with each crop
- Decisions after scenario known (stage 2 decisions):
 - how much wheat/corn to buy in
 - how much of each crop to sell
- Objective (maximise expected profit):
 - maximise expected value of crop sales
 - expected value of grain purchases
 - cost of planting

Stochastic modelling: the Variables

Stage 1 decision variables:

x_w = acres planted with wheat

x_c = acres planted with corn

x_b = acres planted with beets

Stage 2 decision variables:

si_w = tonnes of wheat sold if scenario i occurs

si_c = tonnes of corn sold if scenario i occurs

si_b^H = tonnes of beets sold at high price if scenario i occurs

si_b^L = tonnes of beets sold at low price if scenario i occurs

pi_w = tonnes of wheat purchased if scenario i occurs

pi_c = tonnes of corn purchased if scenario i occurs

Stochastic Modelling: Objective

Objective: Maximize expected profit

expected value of sales

$$\begin{aligned} &= \text{Pr}(\text{scenario 1}) \times (\text{sales under scenario 1}) \\ &\quad + \text{Pr}(\text{scenario 2}) \times (\text{sales under scenario 2}) \\ &\quad + \text{Pr}(\text{scenario 3}) \times (\text{sales under scenario 3}) \end{aligned}$$

$$\begin{aligned} &= \frac{1}{3} (170s_{1_w} + 150s_{1_c} + 36s_{1_b}^H + 10s_{1_b}^L) \\ &\quad + \frac{1}{3} (170s_{2_w} + 150s_{2_c} + 36s_{2_b}^H + 10s_{2_b}^L) \\ &\quad + \frac{1}{3} (170s_{3_w} + 150s_{3_c} + 36s_{3_b}^H + 10s_{3_b}^L) \end{aligned}$$

Stochastic Modelling: Objective

expected value of purchases

$$\begin{aligned} &= \text{Pr}(\text{scenario 1}) \times (\text{purchase under scenario 1}) \\ &\quad + \text{Pr}(\text{scenario 2}) \times (\text{purchase under scenario 2}) \\ &\quad + \text{Pr}(\text{scenario 3}) \times (\text{purchase under scenario 3}) \\ &= \frac{1}{3} (238 p_{1_w} + 210 p_{1_c}) \\ &\quad + \frac{1}{3} (238 p_{2_w} + 210 p_{2_c}) \\ &\quad + \frac{1}{3} (238 p_{3_w} + 210 p_{3_c}) \end{aligned}$$

$$\text{cost of planting} = 150x_w + 230x_c + 260x_b$$

The Farmer's Stochastic LP

max

(expected value of sales)

– (expected value of purchase)

– (cost of planning)

s.t.

$$x_w + x_c + x_b \leq 500$$

$$\text{(wheat yield under scenario } i) x_w + p_{i_w} - s_{i_w} \geq 200, \quad i=1,2,3$$

$$\text{(corn yield under scenario } i) x_c + p_{i_c} - s_{i_c} \geq 240, \quad i=1,2,3$$

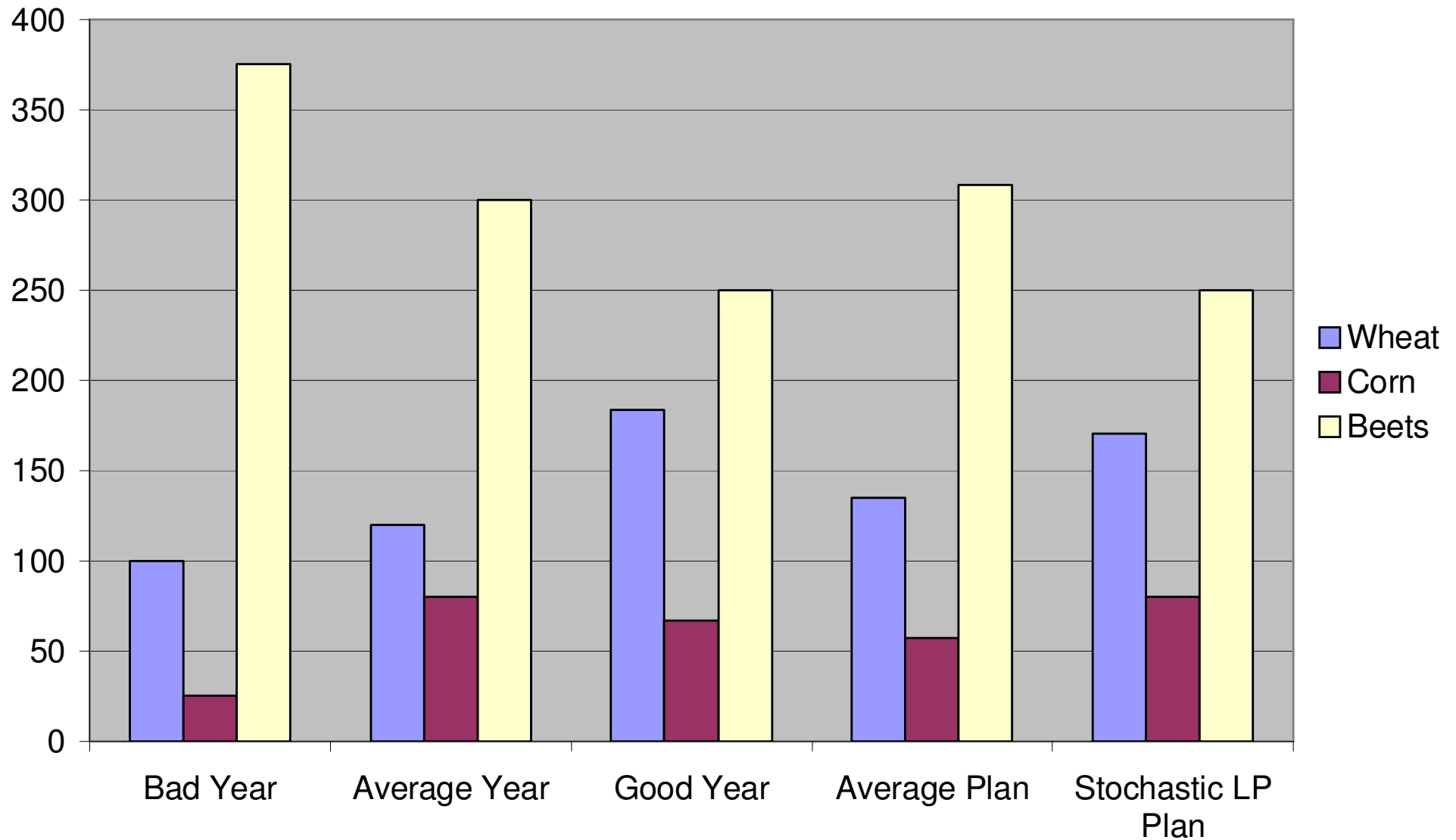
$$s_{i_b}^H + s_{i_b}^L \leq \text{(beet yield under scenario } i) x_b, \quad i=1,2,3$$

$$s_{i_b}^H \leq 6000, \quad i=1,2,3$$

$$x_w, x_c, x_b \geq 0$$

$$p_{i_w}, p_{i_c}, s_{i_w}, s_{i_c}, s_{i_b}^H, s_{i_b}^L \geq 0 \quad i=1,2,3$$

The Stochastic LP Plan



Expected Profit from Stochastic LP

| | | Wheat | Corn | Beets | Overall |
|----------------------------------|--------------|-------|--------|------------------|---------|
| | | | | <= 6000T > 6000T | |
| Yield (T/acre) | | 2.5 | 3 | 20 | |
| Planting cost (\$/acre) | | 150 | 230 | 260 | |
| Selling price (\$/T) | | 170 | 150 | 36 | 10 |
| Additional purchase price (\$/T) | | 238 | 210 | | |
| Acres | | 170 | 80 | 250 | |
| Planting cost (\$) | | 25500 | 18400 | 65000 | |
| Bad Year | Yield (T) | 340 | 192 | 4000 | |
| | Sales (T) | 140 | | 4000 | |
| | Purchase (T) | | 48 | | |
| | Profit (\$) | -1700 | -28480 | 79000 | 48820 |
| Average Year | Yield (T) | 425 | 240 | 5000 | |
| | Sales (T) | 225 | | 5000 | |
| | Purchase (T) | | | | |
| | Profit (\$) | 12750 | -18400 | 115000 | 109350 |
| Good Year | Yield (T) | 510 | 288 | 6000 | |
| | Sales (T) | 310 | 48 | 6000 | |
| | Purchase (T) | | | | |
| | Profit (\$) | 27200 | -11200 | 151000 | 167000 |

Expected profit

$$= (\$48,820 + \$109,350 + \$167,000)/3$$

$$= \$108,390$$

Benefits of Stochastic Model

- Long run average profit if always plan for average scenario = \$107,240
- Long run average profit with stochastic LP plan = \$108,390
- Benefit = \$108,390 - \$107,240 = \$1,150
- Benefits “cascade” with more realistic multistage models, e.g. a 3-period financial planning problem
(see *Introduction to Stochastic Programming*, by J. Birge)

Scenario-based Optimisation in the Literature

- The Peoples Gas Light and Coke Company Plans Gas Supply (Chicago, USA), 1998, by T.W. Knowles & J.P. Wirick

Scenario-based stochastic LP saves \$50 million annually in restructuring gas supply portfolios (see *Interfaces*, Vol 28 (5), 1998)

- <http://www-fp.mcs.anl.gov/otc/Guide/CaseStudies>

- Short-term Financial Planning Under Uncertainty
by J.G. Kallberg, R.W. White and W.T. Ziemba

“We conclude that...the mean model is significantly inferior to the stochastic linear programming formulation. Thus we are able to demonstrate that ignoring the stochastic components...can be very costly without having significant computational savings.”

(see *Management Science*, Vol 28 (6), 1982)