620-362
Applied Operations Research

Quadratic Models

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Quadratic Models

Example:

$$
\begin{cases}
\min \quad 0.2x_1^2 + 0.08x_2^2 + 0.1x_1x_2 \\
s.t. \\
0.14x_1 + 0.11x_2 \geq 120 \\
x_1 + x_2 \leq 1000 \\
x_1, x_2 \geq 0
\end{cases}
$$

Quadratic Models

Matrix notation:

$$\begin{cases} \min \quad \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{bmatrix} 0.2 & 0.05 \\ 0.05 & 0.08 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ s.t. \\ \qquad \begin{bmatrix} 0.14 & 0.11 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} 120 \\ -1000 \end{bmatrix} \\ \qquad x_1, x_2 \geq 0 \end{cases}$$

General Form:

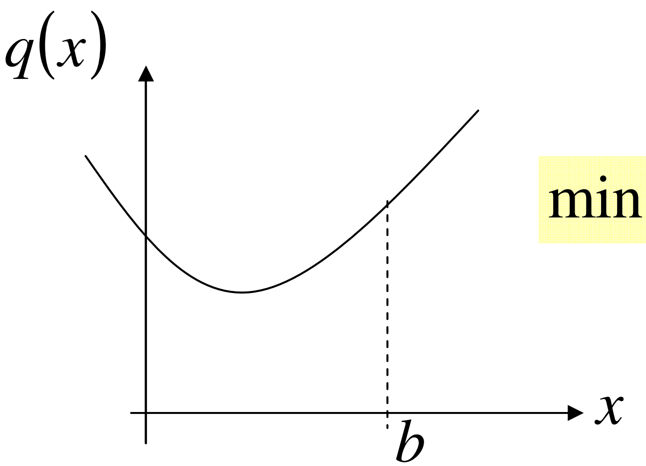
$$\min \quad \mathbf{x}^T \mathbf{Q} \mathbf{x} + \mathbf{c}^T \mathbf{x}$$

$$s.t.$$

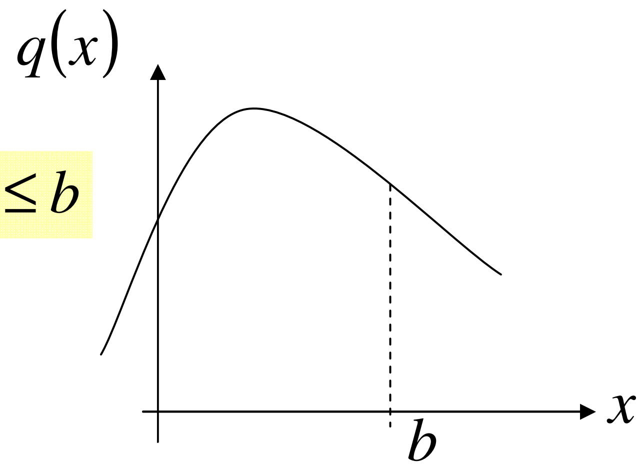
$$\mathbf{A}\mathbf{x} \geq \mathbf{b}$$

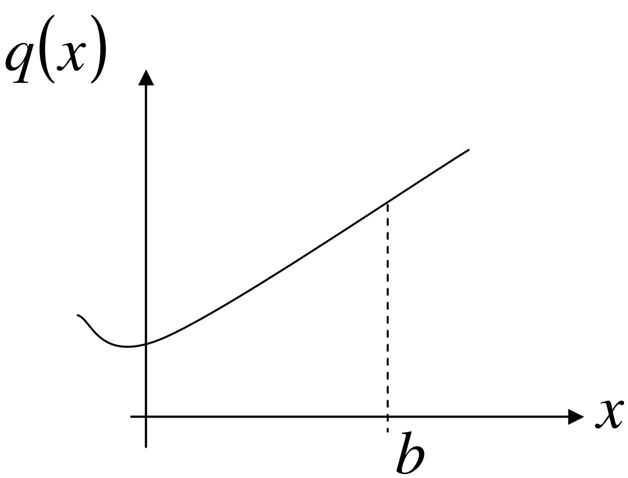
Quadratic Models

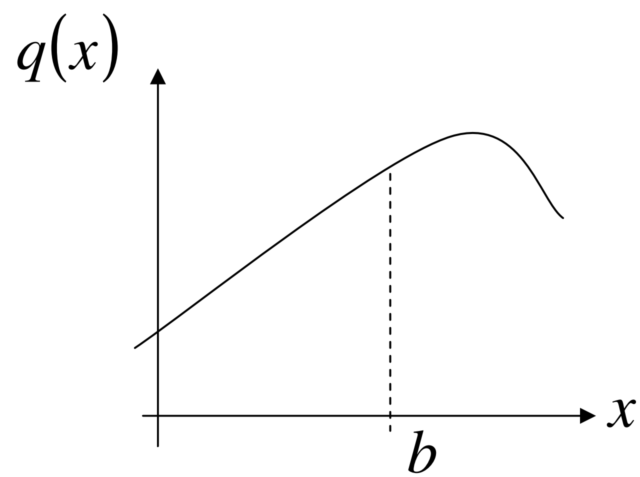
WARNING: unless **Q** is positive definite or positive semidefinite, this is very hard to solve.



$$\min \quad q(x) \quad s.t. \quad 0 \le x \le b$$





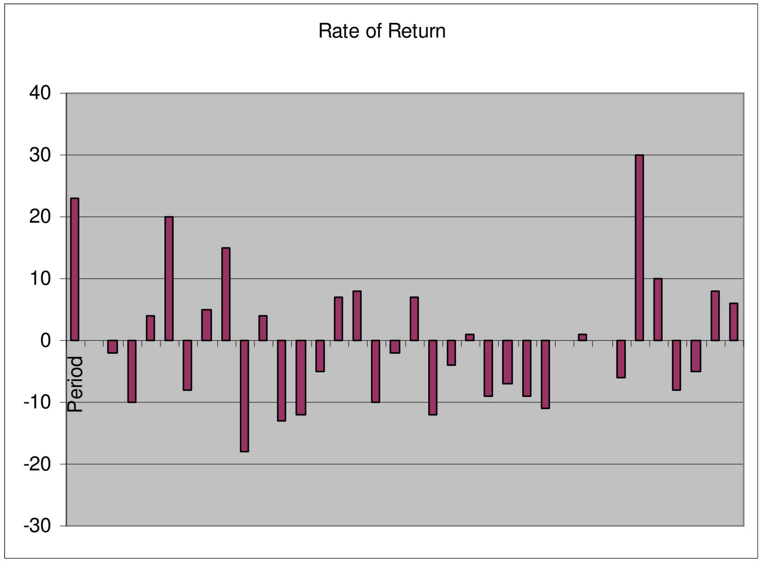


POSITIVE DEFINITE

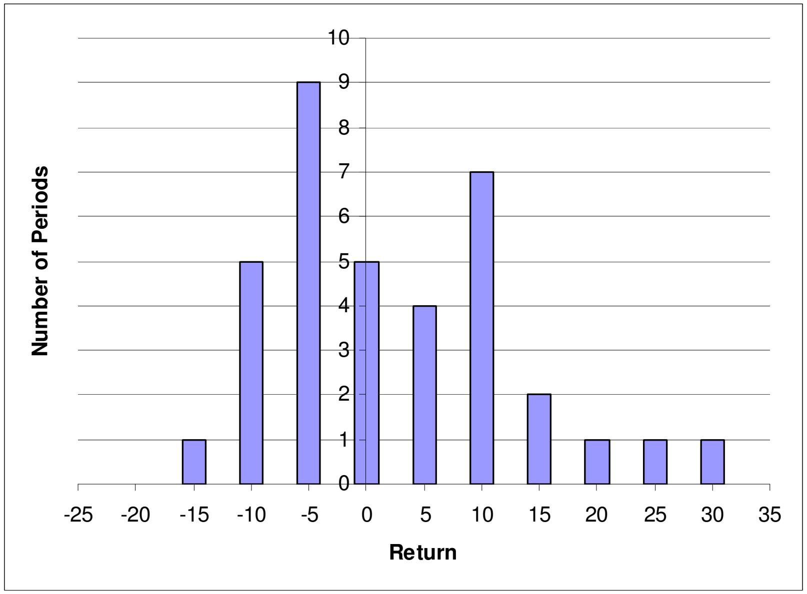
NOT POSITIVE DEFINITE OR POSITIVE SEMI-DEFINITE

Portfolio Analysis

Period	Rate of Return
1	23
2	0
3	-2
4	-10
5	4
6	20
7	-8
8	5
9	15
10	-18
11	4
12	-7
13	4
14	-13
15	5
16	4
17	-1
18	10
19	2
20	-13
21	7
22	-1
23	-5
24	10
25	3
26	7
27	0
28	-12
29	-14
30	-6
31	30
32	10
33	-8
34	-5
35	8
36	6



Rate of return (%) over 36 time periods



Histogram of rate of returns over total period

Portfolio Analysis – Analyse past behaviour

Average return = return we expect from investment.

Variability in asset returns = risk of investment

Let r_i denote return in period *i*, i = 1,…,N

Average (expected) return:

$$E(r) = \frac{r_1 + r_2 + \ldots + r_N}{N} = \frac{\sum_{i=1}^{N} r_i}{N}$$

Riskiness (measured by variance – other measures are also possible)

$$Var(r) = \frac{(r_1 - E(r))^2 + (r_2 - E(r))^2 + \ldots + (r_N - E(r))^2}{N} = \frac{\sum_{i=1}^{N} (r_i - E(r))^2}{N}$$

A Small Example

Return	11	5	-2	8	5	-4	-2	5	11	8
Period	1	2	3	4	5	6	7	8	9	10

Return	-4	-2	5	8	11
Probability	0.1	0.2	0.3	0.2	0.2

Expected return, E(r)

$= (0.1)(-4) + (0.2)(-2) + (0.3)(5) + (0.2)(8) + (0.2)(11)$

$= 4.5$

Var(r)

$= (0.1)(-4-4.5)^2 + (0.2)(-2-4.5)^2 + (0.3)(5-4.5)^2 + (0.2)(8-4.5)^2 + (0.2)(11-4.5)^2$

$= 26.65$

Portfolio Risk and Return

Diversification can reduce risk.

Example: 2 assets

r_i = return of asset 1 in period *i*

s_i = return of asset 2 in period *i*

Investment in asset 1: x ($ or shares)

Investment in asset 2: y ($ or shares)

Return in period *i*

$= xr_i + ys_i$

Portfolio Risk and Return

Expected portfolio return

$$= \frac{1}{N}\sum_{i=1}^{N}\left(xr_i + ys_i\right)$$

$$= \frac{1}{N}\left(x\sum_{i=1}^{N} r_i + y\sum_{i=1}^{N} s_i\right)$$

$$= x\,\frac{\sum_{i=1}^{N} r_i}{N} + y\,\frac{\sum_{i=1}^{N} s_i}{N}$$

$$= xE(r) + yE(s)$$

Portfolio Risk and Return

Portfolio risk

$$= \frac{1}{N}\sum_{i=1}^{N}\left[\left(xr_i + ys_i\right) - \left(xE(r) - yE(s)\right)\right]^2$$

$$= \frac{1}{N}\sum_{i=1}^{N}\left[x\left(r_i - E(r)\right) + y\left(s_i - E(s)\right)\right]^2$$

$$= \frac{1}{N}\sum_{i=1}^{N}\left\{\left[x\left(r_i - E(r)\right)\right]^2 + \left[y\left(s_i - E(s)\right)\right]^2 + 2xy\left(r_i - E(r)\right)\left(s_i - E(s)\right)\right\}$$

$$= x^2\frac{\sum_{i=1}^{N}\left(r_i - E(r)\right)^2}{N} + y^2\frac{\sum_{i=1}^{N}\left(s_i - E(s)\right)^2}{N} + 2xy\frac{\sum_{i=1}^{N}\left(r_i - E(r)\right)\left(s_i - E(s)\right)}{N}$$

$$= x^2\,\mathrm{var}(r) + y^2\,\mathrm{var}(s) + 2xy\,\mathrm{cov}(r,s)$$

Optimal Portfolios

$$\begin{cases} \min \quad \text{riskiness} \\ \text{s.t.} \\ \text{minimum level of return} \\ \text{budget constraint} \end{cases}$$

$$\begin{cases} \min \quad \text{riskiness} - \rho \text{ return} \\ \text{s.t.} \\ \text{budget constraint} \end{cases}$$

In general, we have portfolio with P assets, and would like to invest x_p in asset *p*.

$$\text{Expected portfolio return} = \sum_{p=1}^{P} E\left(r^p\right) x_p$$

$$\text{Riskiness of portfolio} = \sum_{p=1}^{P} \sum_{q=1}^{P} \operatorname{cov}(r^p, r^q) x_p x_q = \mathbf{x}^T \mathbf{Q} \mathbf{x}$$

where $\mathbf{x}$ is vector of investments and $\mathbf{Q}$ is the covariance matrix.

Small Example

Return Asset 1 (%)	11	5	-2	8	5	-4	-2	5	11	8
Return Asset 2 (%)	15	3	3	-3	7	3	7	15	-3	9
Period	1	2	3	4	5	6	7	8	9	10

$E(R^1) = 4.5$

$Var(R^1) = 26.25$

$E(R^2) = 5.6$

$Var(R^2) = 36.04$

$Cov(R^1,R^2)$

$= 1/10 [(11-4.5)(15-5.6) + (5-4.5)(3-5.6)+\ldots+(8-4.5)(9-5.6)]$

$= 2.1$

$$\sum_{p=1}^{P}\sum_{q=1}^{P} \text{cov}(r^p, r^q) x_p x_q = \mathbf{x}^T \mathbf{Q} \mathbf{x}$$

Small Example

Invest x_1 in asset 1

Invest x_2 in asset 2

Expected portfolio return

$= 4.5x_1 + 5.6x_2$

$$\sum_{p=1}^{P}\sum_{q=1}^{P}\operatorname{cov}(r^p, r^q)x_p x_q = \mathbf{x}^T\mathbf{Q}\mathbf{x}$$

Riskiness of portfolio =

$= 26.25x_1^2 + 36.04x_2^2 + 2(2.1)x_1x_2$

$$= \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{bmatrix} 26.25 & 2.1 \\ 2.1 & 36.04 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Small Example

Budget = \$2,000 (spend all)

Minimum return required = 5%

$$
\begin{cases}
\min \quad \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{bmatrix} 26.25 & 2.1 \\ 2.1 & 36.04 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
s.t. \\
4.5x_1 + 5.6x_2 \geq 10000 \\
x_1 + x_2 = 2000 \\
x_1, x_2 \geq 0
\end{cases}
$$

$$
\frac{(4.5x_1 + 5.6x_2)/100}{2000} \geq \frac{5}{100}
$$

Solving QP with XpressMP

```
model "Portfolio Optimisation"
    uses "mmxprs", "mmquad"; !gain access to the Xpress-Optimizer solver

    !sample declarations section
    parameters
        DATAFILE = "portfolio.txt"
        T = 10
        N = 2
    end-parameters

    declarations
        PERIODS = 1..T
        ASSETS = 1..N

        Budget: real
        MinReturn: real

        Return: array(ASSETS,PERIODS) of real
        ExpectedReturn: array(ASSETS) of real
        Covariance: array(ASSETS,ASSETS) of real

        x: array(ASSETS) of mpvar
    end-declarations

    initialisations from DATAFILE
        Budget
        MinReturn
        Return
    end-initialisations

    forall(p in ASSETS) do
        ExpectedReturn(p):= sum(t in PERIODS) Return(p,t)/getsize(PERIODS)
    end-do

    forall(p in ASSETS, q in ASSETS) do
        Covariance(p,q):= (sum(t in PERIODS) (Return(p,t) - ExpectedReturn(p))*(Return(q,t) - ExpectedReturn(q)))/getsize(PERIODS)
    end-do

    sum(p in ASSETS) ExpectedReturn(p) * x(p) >= MinReturn

    sum(p in ASSETS) x(p) = Budget

    Risk:= sum(p in ASSETS, q in ASSETS) Covariance(p,q) * x(p) * x(q)

    minimise(Risk)
```

$$\begin{cases} \min \quad \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{bmatrix} 26.25 & 2.1 \\ 2.1 & 36.04 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ s.t. \\ 4.5x_1 + 5.6x_2 \geq 10000 \\ x_1 + x_2 = 2000 \\ x_1, x_2 \geq 0 \end{cases}$$

Portfolio Optimisation: Xpress^{MP} Example

An investor is evaluating ten different securities (`shares').

He estimates the return on investment for a period of one year.

He further wishes to invest at least half of his capital in North-American shares and at most a third in shares.

How should the capital be divided among the shares to minimize the risk whilst obtaining a certain target yield?

The investor adopts the **Markowitz** idea of getting estimates of the variance/covariance matrix of estimated returns on the securities.

Portfolio Optimisation: Xpress^{MP} Example

Given

I^{max} = max. investment per share (%)

I^{min} = min. investment into North American share (%)

S = set of all shares

A = set of North American shares

T = target yield (%)

R_p = estimated return in investment for share *p* (%)

Cov = covariance matrix of estimated return

Variable:

x_p = fraction of capital invested in share *p*

Portfolio Optimisation: XpressMP Example

$$\min \quad \sum_{p\in S}\sum_{q\in S} \operatorname{cov}(p,q)x_p x_q$$

$$s.t.$$

$$\sum_{p\in A} x_p \geq I^{\min}$$

$$\sum_{p\in S} x_p = 1$$

$$x_p \leq I^{\max}, \quad \forall p \in S$$

$$\sum_{p\in S} R_p x_p \geq T$$

$$x_p \geq 0, \quad \forall p \in S$$

Portfolio Optimisation: Xpress^{MP} Example

Xpress^{MP} demonstration