

**620-362**  
**Applied Operations Research**  
**Piecewise Linear Models**

**Department of Mathematics and Statistics**  
**The University of Melbourne**

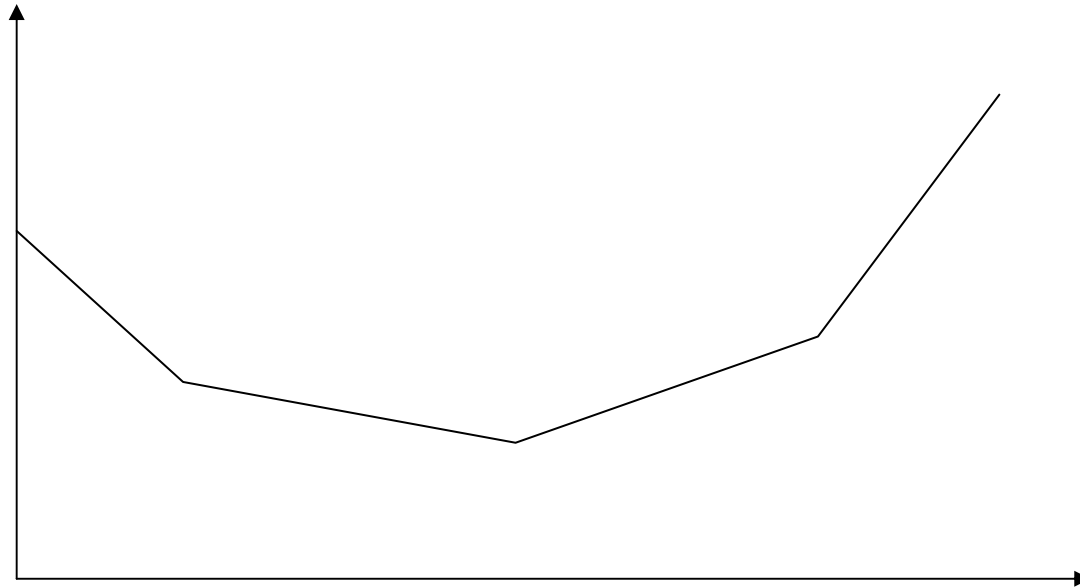
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# Uses

1. Penalties on “soft constraints”
2. Dealing with infeasibility
3. Reversible activities
4. Modelling piecewise linear costs, e.g. overtime, tax
5. Approximating nonlinear functions



# Penalties on Soft Constraints

**Example 1:** Total hours used in production is a soft constraint, with a penalty on profit of \$100 per hour used over limit of  $T$  hours.

Let variable

$$x_p = \# \text{ units product } p \text{ made}$$

Let

$$H_p = \# \text{ hours/unit used for product } p$$

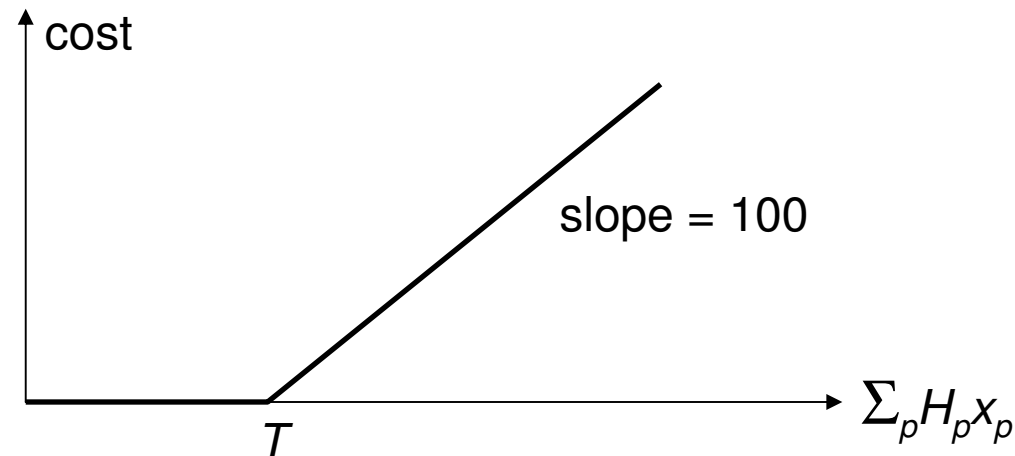
“Hard” constraint:

$$\sum_p H_p x_p \leq T$$

“Soft” constraint:

$$\text{if } \sum_p H_p x_p > T, \text{ pay } 100(\sum_p H_p x_p - T)$$

# Penalties on Soft Constraints



To model as an LP, introduce new variable

$$y = \text{excess hours used}$$

via

$$y \geq \sum_p H_p x_p - T, y \geq 0$$

with objective function

$$\min \dots + 100y + \dots$$

# Penalties on Soft Constraints

**Example 2:** Time windows of 12pm to 2pm for delivery to a customer is a soft constraint with a penalty of \$20 per hour early and \$50 per hour late.

Let

$t$  = time delivery made

“Hard” constraint:

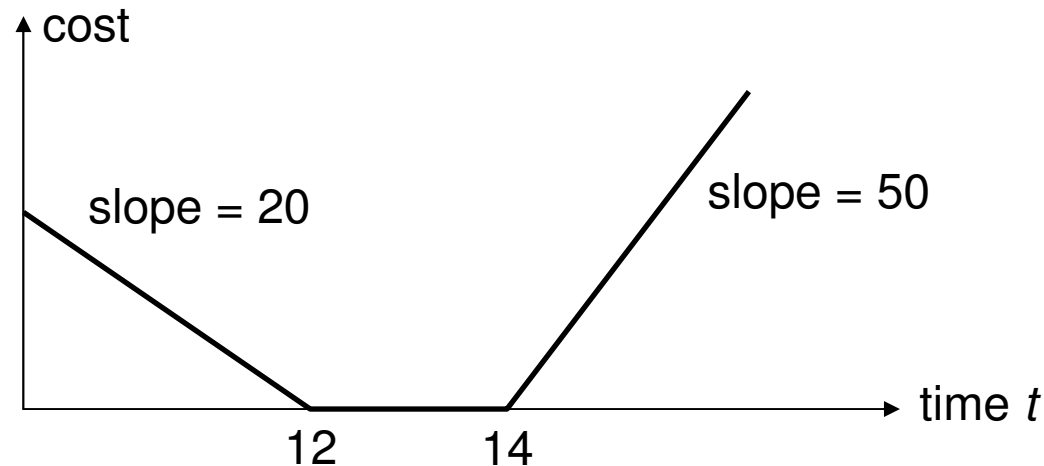
$$12 \leq t \leq 14$$

“Soft” constraint:

if  $t < 12$  then pay  $20(12 - t)$

if  $t > 14$  then pay  $50(t - 14)$

# Penalties on Soft Constraints



To model as an LP, introduce two new variables

$y = \#$  hours early

$z = \#$  hours late

via

$$y \geq 12 - t, y \geq 0$$

$$z \geq t - 14, z \geq 0$$

with objective function

$$\min \dots + 20y + 50z + \dots$$

# Dealing with Infeasibility

**Example:** Production capacity may be insufficient to meet minimum customer demands.

Solver will just report “problem infeasible”

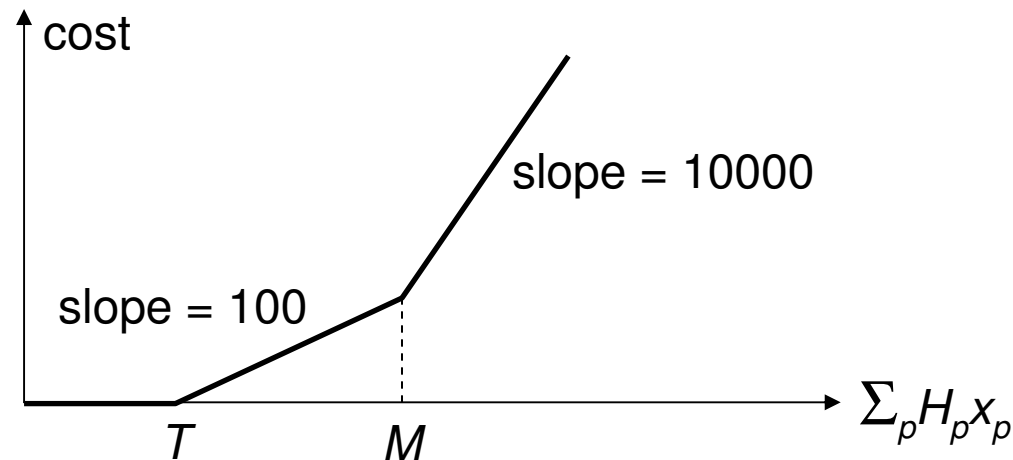
- which demands are excessive?
- where would greater production time be needed?

Answer: convert some constraints to “soft” constraints with very large penalties on violation.

From Example 1, suppose production cannot exceed a limit  $M > T$ , but problem is infeasible.

# Dealing with Infeasibility

Convert  $\sum_p H_p x_p \leq M$  to soft constraint with penalty 10,000.



To model this as an LP, introduce two new variables

$y_1 = \#$  hours used in excess of  $T$

$y_2 = \#$  hours used in excess of  $M$

via

$$y_1 \geq \sum_p H_p x_p - T, \quad y_1 \geq 0$$

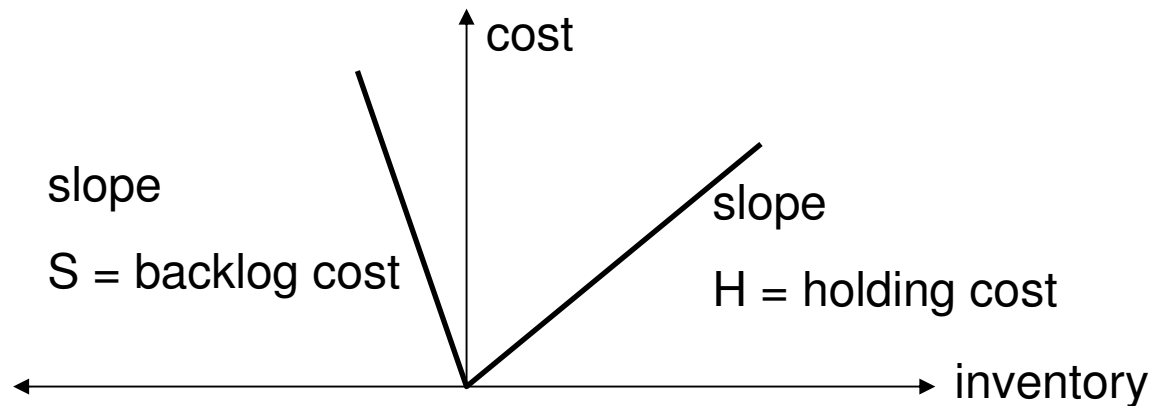
$$y_2 \geq \sum_p H_p x_p - M, \quad y_2 \geq 0$$

and objective

$$\min \dots 100y_1 + (10000 - 100)y_2 + \dots$$

# Reversible Activities

**Example 3:** Negative inventory can be viewed as backlog



Variable  $x_t = \#$  products made in time period  $t$ ,  $\forall t = 1, \dots, T$

Variable  $i_t =$  inventory held through period  $t$ ,  $\forall t = 1, \dots, T$

Have data

$D_t =$  demand in period  $t$

$C =$  cost per unit produced

$H =$  holding cost per unit per period

$S =$  backlog cost per unit per period

$I_0 =$  initial level of inventory at start of planning period

# Reversible Activities

## Constraints

$$i_t = i_{t-1} + x_t - D_t, \quad \forall t = 1, \dots, T$$

Cost in period  $t$  is:

$$\begin{cases} Cx_t + Hi_t, & \text{if } i_t \geq 0 \\ Cx_t - Si_t, & \text{if } i_t < 0 \end{cases}$$

To model linearly, introduce new variables

$$y_t = \begin{cases} i_t, & \text{if } i_t \geq 0 \\ 0, & \text{if } i_t < 0 \end{cases} = i_t^+ \quad z_t = \begin{cases} -i_t, & \text{if } i_t < 0 \\ 0, & \text{if } i_t \geq 0 \end{cases} = -i_t^+$$

# Reversible Activities

via

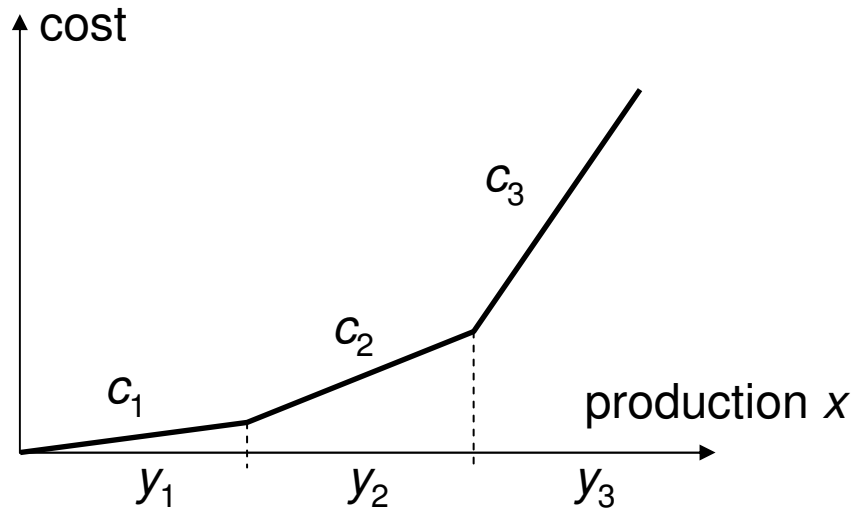
$$y_t \geq i_t, \quad y_t \geq 0, \quad \forall t = 1, \dots, T$$

$$z_t \geq -i_t, \quad z_t \geq 0, \quad \forall t = 1, \dots, T$$

with objective

$$\min \sum_{t=1}^T Cx_t + Hy_t + Sz_t$$

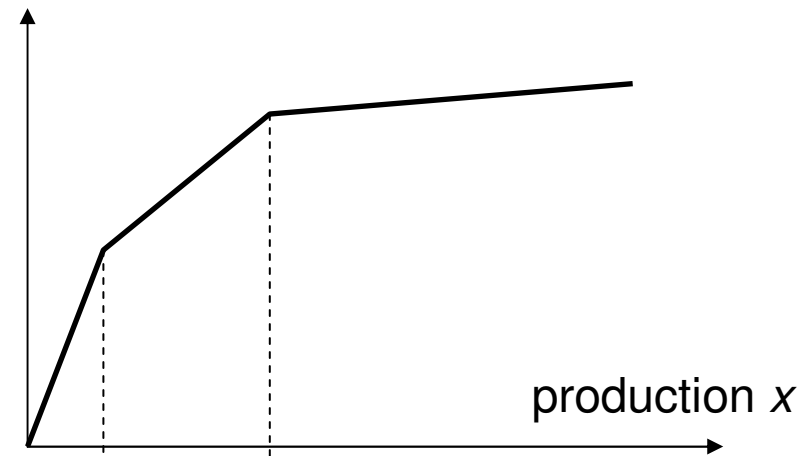
# Piecewise Linear LP Models



e.g. overtime costs, tax effects

minimise convex/maximise concave  
piecewise linear function

EASY! LP-model



e.g. economies of scale

minimise non-convex or  
concave/maximise non-concave or  
convex piecewise linear function

HARD: Integer Programming Model

# Production Scheduling

Machine	Max. # bottles able to produce per month	Labour utilization per bottle per month
1	$a_1$	$b_1$
2	$a_2$	$b_2$

Planning period:  $t = 1, 2, \dots, 12$

Demand for bottles:  $d_t$        $t = 1, 2, \dots, 12$

Labour availability:  $L_t$        $t = 1, 2, \dots, 12$

Initial inventory:  $I_0$

**Objective:** Determine production levels on each machine in each month so as to meet demand, and minimise maximum monthly fluctuation in labour utilization.

# Production Scheduling

Primary decision variables:

$x_{1t}$  = # bottles produced on machine 1 in month  $t$ ,

$x_{2t}$  = # bottles produced on machine 2 in month  $t$ ,

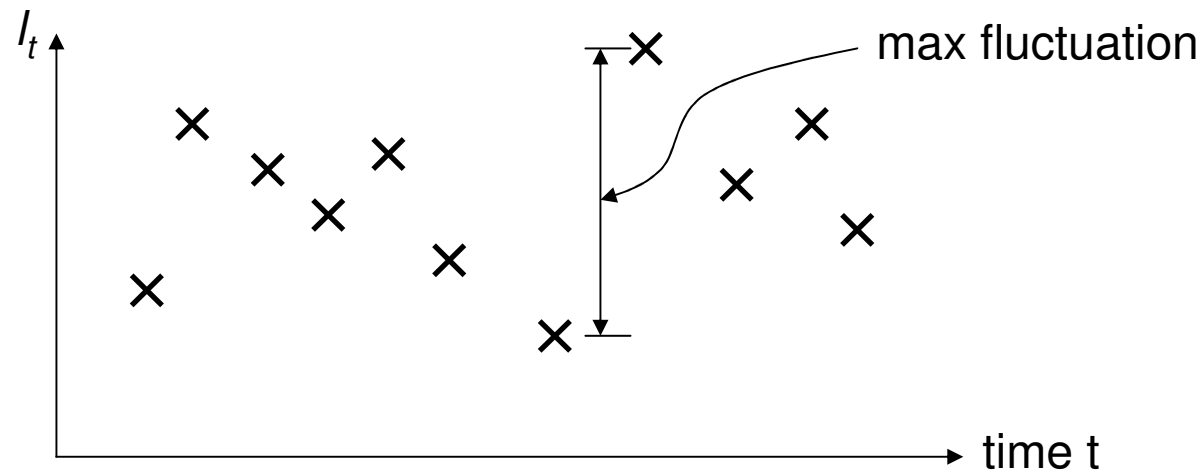
$t = 1, 2, \dots, 12$

Inventory variables:

$y_t$  = # bottles in inventory at end of month  $t$

Labour utilization variables:

$l_t$  = labour utilization in month  $t$



# Production Scheduling

$$\min \max_{t=2,\dots,12} |l_t - l_{t-1}|$$

$$s.t. \quad l_t = b_1 x_{1t} + b_2 x_{2t}, \quad \forall t = 1, \dots, 12$$

$$y_1 = x_{11} + x_{21} + I_0 - d_1$$

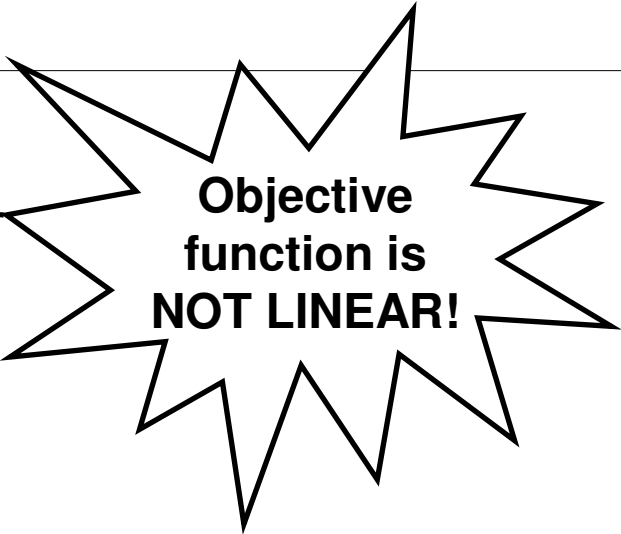
$$y_t = x_{1t} + x_{2t} + y_{t-1} - d_t, \quad \forall t = 2, \dots, 12$$

$$0 \leq x_{1t} \leq a_1, \quad \forall t = 1, \dots, 12$$

$$0 \leq x_{2t} \leq a_2, \quad \forall t = 1, \dots, 12$$

$$0 \leq l_t \leq L_t, \quad \forall t = 1, \dots, 12$$

$$y_t \geq 0, \quad \forall t = 1, \dots, 12$$



**Objective  
function is  
NOT LINEAR!**

# Production Scheduling

Introduce new variable:  $m = \max_{t=2,\dots,12} |l_t - l_{t-1}|$

i.e.

$$\left\{ \begin{array}{l} \min \\ s.t. \end{array} \right. \begin{array}{l} m \\ m \geq |l_t - l_{t-1}|, \quad \forall t = 2, \dots, 12 \\ l_t = b_1 x_{1t} + b_2 x_{2t}, \quad \forall t = 1, \dots, 12 \\ \vdots \\ etc \end{array}$$



Still NOT  
LINEAR!

# Production Scheduling

Consider this:  $w \geq |v| \iff w \geq v \text{ and } w \geq -v$

LP Model:

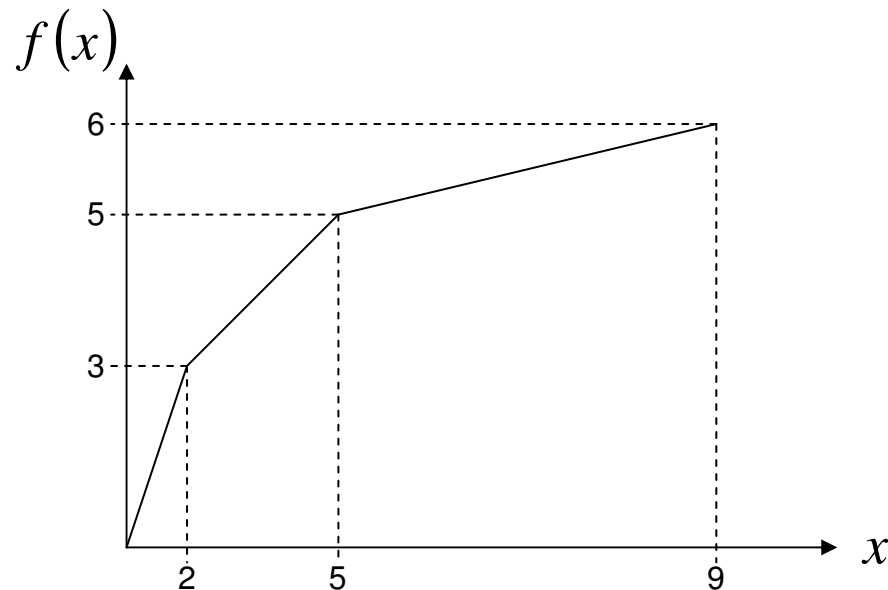
$$\left\{ \begin{array}{l} \min \\ s.t. \end{array} \right. \left. \begin{array}{l} m \\ m \geq l_t - l_{t-1} \\ m \geq l_{t-1} - l_t \end{array} \right\}, \quad \forall t = 2, \dots, 12$$
$$l_t = b_1 x_{1t} + b_2 x_{2t}, \quad \forall t = 1, \dots, 12$$
$$\vdots$$

*etc*

# Max. a Piecewise Linear Concave Function

## Example:

$$\left\{ \begin{array}{l} \max \quad f(x_1) + 2x_2 \\ \text{s.t.} \quad x_1 + 3x_2 \leq 15 \\ \quad \quad x_1 + x_2 \leq 9 \\ \quad \quad x_1, x_2 \geq 0 \end{array} \right. \quad \text{where } f(x) = \begin{cases} \frac{3}{2}x, & 0 \leq x \leq 2 \\ \frac{1}{3}(2x+5), & 2 < x \leq 5 \\ \frac{1}{4}(x+15), & 5 < x \leq 9 \end{cases}$$



# Max. a Piecewise Linear Concave Function

We can re-model this problem as an LP by replacing

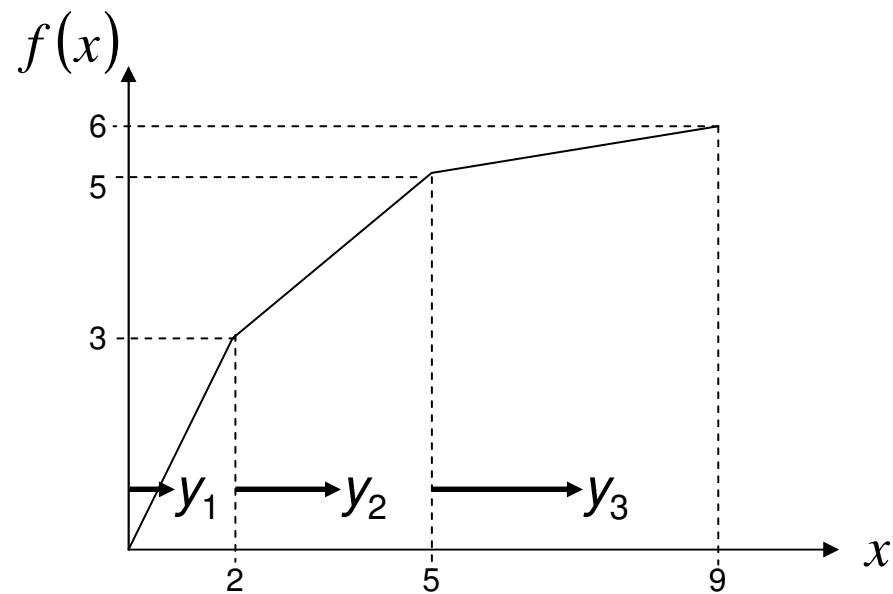
$$x_1 = y_1 + y_2 + y_3$$

in all constraints, where

$$0 \leq y_1 \leq 2$$

$$0 \leq y_2 \leq 3$$

$$0 \leq y_3 \leq 4$$



and replacing  $f(x_1)$  in the objective by

$$3/2y_1 + 2/3y_2 + 1/4y_3$$

# Max. a Piecewise Linear Concave Function

New model is:

$$\left\{ \begin{array}{l} \max \quad \frac{3}{2} y_1 + \frac{2}{3} y_2 + \frac{1}{4} y_3 + 2x_2 \\ \text{s.t.} \quad y_1 + y_2 + y_3 + 3x_2 \leq 15 \\ \quad \quad y_1 + y_2 + y_3 + x_2 \leq 9 \\ \quad \quad 0 \leq y_1 \leq 2 \\ \quad \quad 0 \leq y_2 \leq 3 \\ \quad \quad 0 \leq y_3 \leq 4 \\ \quad \quad x_2 \geq 0 \end{array} \right.$$

# Max. a Piecewise Linear Concave Function

For the new model to be correct, we need the optimal solution  $(\mathbf{y}^*, x^*_2)$  to satisfy:

1. if  $y^*_1 < 2$  then  $y^*_2 = y^*_3 = 0$ , and
2. if  $y^*_2 < 3$  then  $y^*_3 = 0$

Can we be sure of this?

We will show by contradiction.

## Max. a Piecewise Linear Concave Function

Suppose  $(\mathbf{y}^*, x_2^*)$  is optimal,

and  $y_1^* < 2$  and  $y_2^* > 0$ .

Set  $\varepsilon = \min\{y_2^*, 2 - y_1^*\}$ . Note  $\varepsilon > 0$ .

Set  $\hat{y}_1 = y_1^* + \varepsilon$

$$\hat{y}_2 = y_2^* - \varepsilon$$

$$\hat{y}_3 = y_3^*$$

$$\hat{x}_2 = x_2^*$$

Now  $\hat{y}_1 + \hat{y}_2 + \hat{y}_3 = y_1^* + \varepsilon + y_2^* - \varepsilon + y_3^* = y_1^* + y_2^* + y_3^*$

So  $(\hat{\mathbf{y}}, \hat{x}_2)$  must be feasible.

# Max. a Piecewise Linear Concave Function

The objective value of  $(\hat{y}, \hat{x}_2)$  is :

$$\frac{3}{2}(y_1^* + \varepsilon) + \frac{2}{3}(y_2^* - \varepsilon) + \frac{1}{4}y_3^* + 2x_2^* = \frac{3}{2}y_1^* + \frac{2}{3}y_2^* + \frac{1}{4}y_3^* + 2x_2^* + \frac{5}{6}\varepsilon$$

> objective value of  $(y^*, x_2^*)$

Therefore,  $(y^*, x_2^*)$  is not optimal!

This is a contradiction! So it must be that if  $y_1^* < 2$  then  $y_2^* = 0$ .

The other properties needed can be shown to hold by similar arguments.

**Exercise:** Show that if  $y_1^* < 2$  then  $y_3^* = 0$ , and that if  $y_2^* < 3$  then  $y_3^* = 0$ .

THUS THE NEW LP MODEL IS VALID!

# Max. a Piecewise Linear Concave Function

The LP model has optimal solution

$$y^*_1 = 2, \quad y^*_2 = 3, \quad y^*_3 = 0, \quad x^*_2 = 3^{1/3}$$

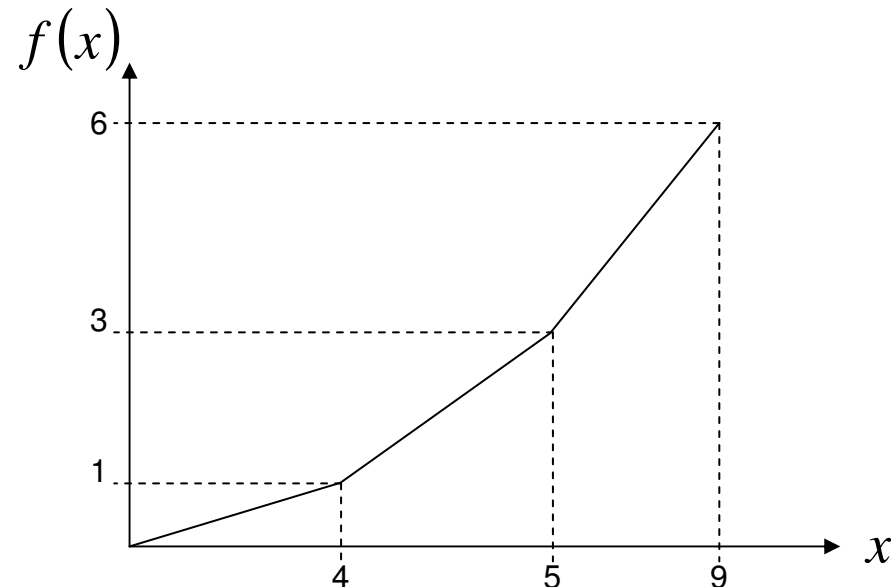
i.e.

$$x^*_1 = 5, \quad x^*_2 = 3^{1/3}$$

is the optimal solution to the original problem.

# Max. a Piecewise Linear Concave Function

What if  $f$  was conVEX instead?



If we do the same kind of LP model with  $x_1 = y_1 + y_2 + y_3$  in the constraints where

$$0 \leq y_1 \leq 4, \quad 0 \leq y_2 \leq 3 \text{ and } 0 \leq y_3 \leq 4$$

we get the optimal LP solution to be

$$y_1^* = 0, \quad y_2^* = 3, \quad y_3^* = 2, \quad x_1^* = 3^{1/3}$$

with LP value  $11^{2/3}$ .

## Max. a Piecewise Linear Concave Function

But then  $x^*_1 = 5$ ,  $x^*_2 = 3\frac{1}{3}$  which has objective value  $8\frac{1}{3}$  in the original problem!

The LP model is not using the correct objective!

**Exercise:** Check this by solving the LP.

**Hard exercise:** What is the optimal solution of the original problem?

# Max. a Piecewise Linear Concave Function

In general, a problem maximising a convex function CANNOT be re-modelled as an LP.

Similarly minimising convex function which are piecewise linear can be solved using an LP model, but minimising concave function CANNOT be.

For general functions:

	<b>Minimising</b>	<b>Maximising</b>
CONVEX	EASY	HARD
CONCAVE	HARD	EASY
NEITHER	HARD	HARD