620-362
Applied Operations Research

Other MIP Problems

Department of Mathematics and Statistics
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Some contents of this presentation are adapted from year 2005 course notes for 620-362 Applied Operations Research, Department of Mathematics and Statistics, The University of Melbourne (compiled by Prof Natashia Boland and Dr Renata Sotirov)

Motivation

A quick MIP modelling revision, applicable to

- Assignments
- Industrial project
- Mid-semester test

via examples.

Structure:

- Decomposing the "story"
- Adding complexity to a "basic" problem

Ambulance Service Improvement

The Utopian Ambulance Service (UAS) is the main provider of ambulance services in the state of Peaceful. The state of Peaceful has **eight districts** in total, called Atlantis-1, Atlantis-2, Atlantis-3, Atlantis-4, Atlantis-5, Atlantis-6, Atlantis-7 and Atlantis-8.

Currently, UAS has **ten ambulance service centres** (ASC), providing services to the eight different districts.

UAS wishes to improve their services by **purchasing new ambulances**, and if needed, will **upgrade their facilities to accommodate the increase in the number of ambulances**.

UAS has allocated $1,000,000 for this purpose. The penalty for not meeting the **budget** is $1 per dollar exceeding the budgeted amount.

Ambulance Service Improvement

Each district will be served by all ASC within a 10km radius from the centre of the district, as listed in Table 1. Due to political reasons, **at most one ASC in each district, i.e. within a 10km limit from the center of the district, can be upgraded**. UAS will sign a quality-of-service agreement with each district, stating that UAS's average response time for service within that district should not exceed the Average Response Time Guarantee (ARTG) specified in Table 1.

District	***ASC within 10km radius***	***Average Response Time Guarantee (ARTG) (minutes)***	***Penalty per minute exceeding ARTG ($)***
Atlantis-1	UC-1, UC-2	7	15000
Atlantis-2	UC-2, UC-3	5	12000
Atlantis-3	UC-4, UC-6	6	13000
Atlantis-4	UC-1, UC-10	8	18000
Atlantis-5	UC-4, UC-5, UC-9	5	9000
Atlantis-6	UC-5, UC-8	5	13000
Atlantis-7	UC-7, UC-9, UC-10	7	14000
Atlantis-8	UC-3, UC-6	6	12000

Ambulance Service Improvement

The response time of an ASC$_j$ can be estimated by

$$R_j^{ASC} = \max\left(5, 10\alpha_j - \frac{5}{\alpha_j} n_j\right)$$

for $n_j > 0$, where n_j is the number of ambulances at ASC$_j$ and α_j is the number of districts ASC$_j$ is serving.

$$\overline{R}_i^{District} = \frac{\sum_{j \in S_i} R_j^{ASC}}{|S_i|}$$

where S_i is the set of all ASC's within a 10km radius from the centre of district *i*.

Ambulance Service Improvement

Whenever an ASC is upgraded, a fixed cost will be incurred. The number of ambulances currently stationed at the ASC, the capacity when the facility is upgraded and the fixed cost of upgrade are shown for each ASC in Table 2. The cost of a new ambulance is $70,000 each. All ASC's are currently at their full capacity.

ASC	***Number of ambulances***	***Capacity when facility is upgraded (number of ambulances)***	***Fixed Cost when upgrading is performed ($)***
UC-1	3	10	100000
UC-2	4	20	120000
UC-3	3	18	160000
UC-4	4	17	180000
UC-5	3	12	130000
UC-6	5	13	100000
UC-7	2	20	180000
UC-8	4	18	150000
UC-9	5	20	180000
UC-10	2	11	120000

Ambulance Service Improvement

Given:

d	be the number of districts
m	be the number of ASC's
S_i	be the set of ASC within 10km radius from the center district *i*.
n	be the maximum number of upgrades for any district
a_j	be the number of ambulances currently available at ASC *j*
$\overline{C}_j$	be the capacity of the upgraded ASC *j*
c	be the cost of purchasing an ambulance
f_j	be the cost of upgrading ASC *j*
T_i	be the ARTG at district *i*
p_i	be the penalty per minute exceeding ARTG, T_i at district *i*
B	be ASC's budget
$\overline{p}$	be the penalty per dollar for exceeding B

Ambulance Service Improvement

Decisions:

- # new ambulances -> upgrade?

Also want to measure:

- response time
- budget

$$y_j = \begin{cases} 1, & if \ ASC \ j \ is \, upgraded \\ 0, & otherwise \end{cases}$$

x_j is the number of new ambulances purchased at ASC *j*

r_i^+ is the average response time exceeding ARTG in district *i*

R_j be the response time of ASC *j*

b^+ is the amount exceeding budget *B*

Ambulance Service Improvement

Response constraints:

$$R_j^{ASC} = \max\left(5, 10\alpha_j - \frac{5}{\alpha_j} n_j\right)$$

Actual response:

$$R_j \geq 5, \qquad j = 1 \ldots m$$

$$R_j \geq \left[10\alpha_j - \frac{5}{\alpha_j}\left(x_j + a_j\right)\right], \qquad j = 1 \ldots m$$

Average response requirement:

$$\overline{R}_i^{District} = \frac{\sum_{j \in S_i} R_j^{ASC}}{|S_i|}$$

$$\frac{\sum_{j \in S_i} R_j}{|S_i|} - T_i \leq r_i^+, \qquad i = 1 \ldots d$$

Ambulance Service Improvement

Max # upgrades:

$$\sum_{j \in S_i} y_j \leq n, \qquad i = 1 \ldots d$$

ASC capacity limit:

$$x_j \leq y_j \left(\overline{C}_j - a_j \right), \qquad j = 1 \ldots m$$

Budget:

$$\sum_{j=1}^{m} \left(c x_j + f_j y_j \right) - B \leq b^+$$

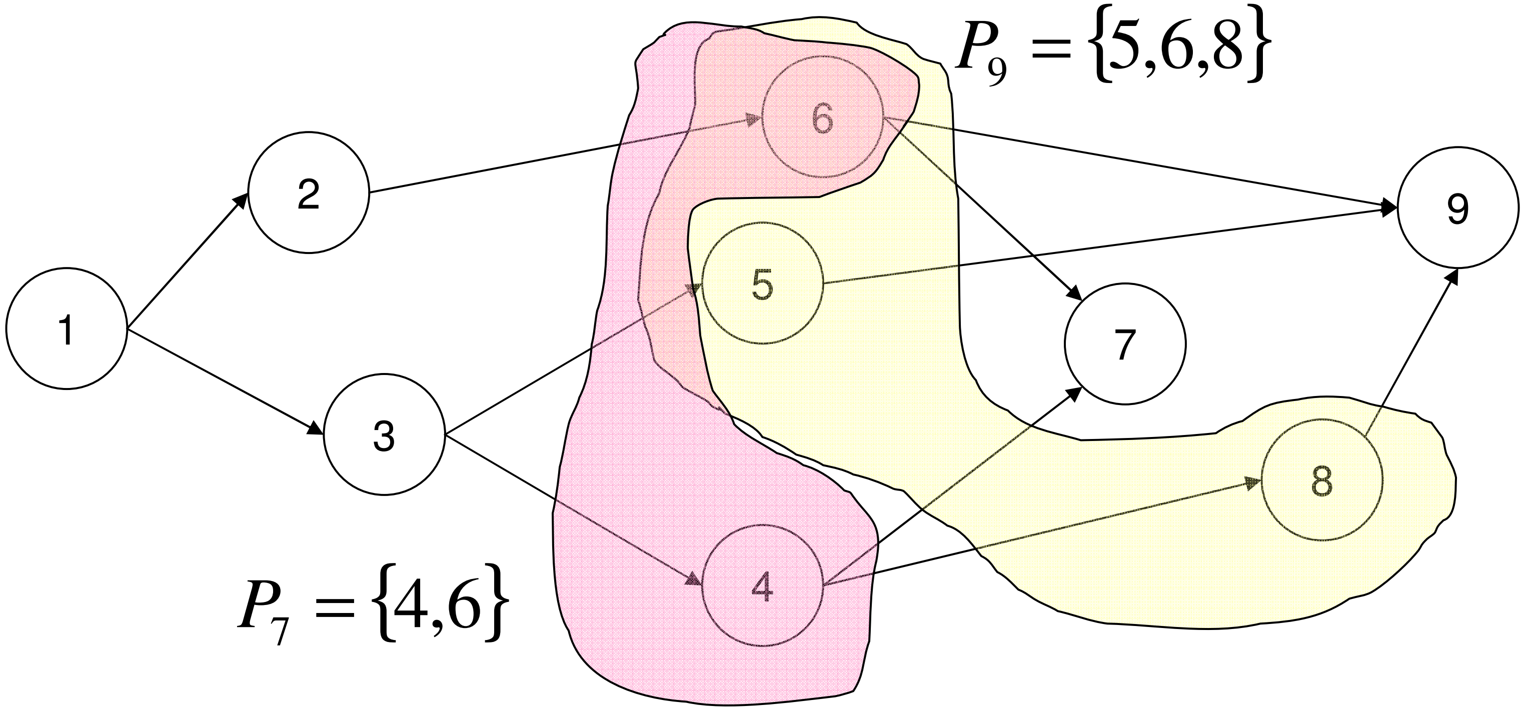
Objective function:

$$\text{Minimise } \overline{p} b^+ + \sum_{i=1}^{d} p_i r_i^+$$

Project Management

Given

- S = set of all tasks
- D_i = duration of task *i*
- P_i = set of tasks **immediately** preceding task *i*



Project Management

Objective 1:

Minimise the maximum completion time of all tasks.

Variables:

t_i = start time of task *i*

$$t_i \geq \max_{j \in P_i} \left(t_j + D_j \right)$$

$$\Downarrow$$

$$t_i \geq t_j + D_j, \quad \forall j \in S_i$$

Project Management

$$\min \quad \max_{i \in S} \left(t_i + D_i \right)$$

$$s.t.$$

$$t_i \geq t_j + D_j, \quad \forall i \in S,\ j \in P_i$$

$$t_i \geq 0, \quad \forall i \in S$$

$$\Downarrow$$

$$\min \quad z$$

$$s.t.$$

$$z \geq t_i + D_i, \quad \forall i \in S$$

$$t_i \geq t_j + D_j, \quad \forall i \in S,\ j \in P_i$$

$$t_i \geq 0, \quad \forall i \in S$$

Project Management

Objective 2:

Minimise total *crashing* (reducing task duration) cost such that project is completed within time T.

Given

C_i = cost of crashing task *i*

Variables

x_i = amount by which duration of task *i* is reduced

Project Management

$$\min \quad \sum_{i \in S} C_i x_i$$

$$s.t.$$

$$t_i + D_i - x_i \leq T, \quad \forall i \in S$$

$$t_i \geq t_j + D_j - x_j, \quad \forall i \in S, j \in P_i$$

$$x_i \leq D_i, \quad \forall i \in S$$

$$t_i, x_i \geq 0, \quad \forall i \in S$$

Project Management

Now suppose there is a Human Resource aspect to this problem (with Objective 2).

Each worker can only carry out a set of tasks.

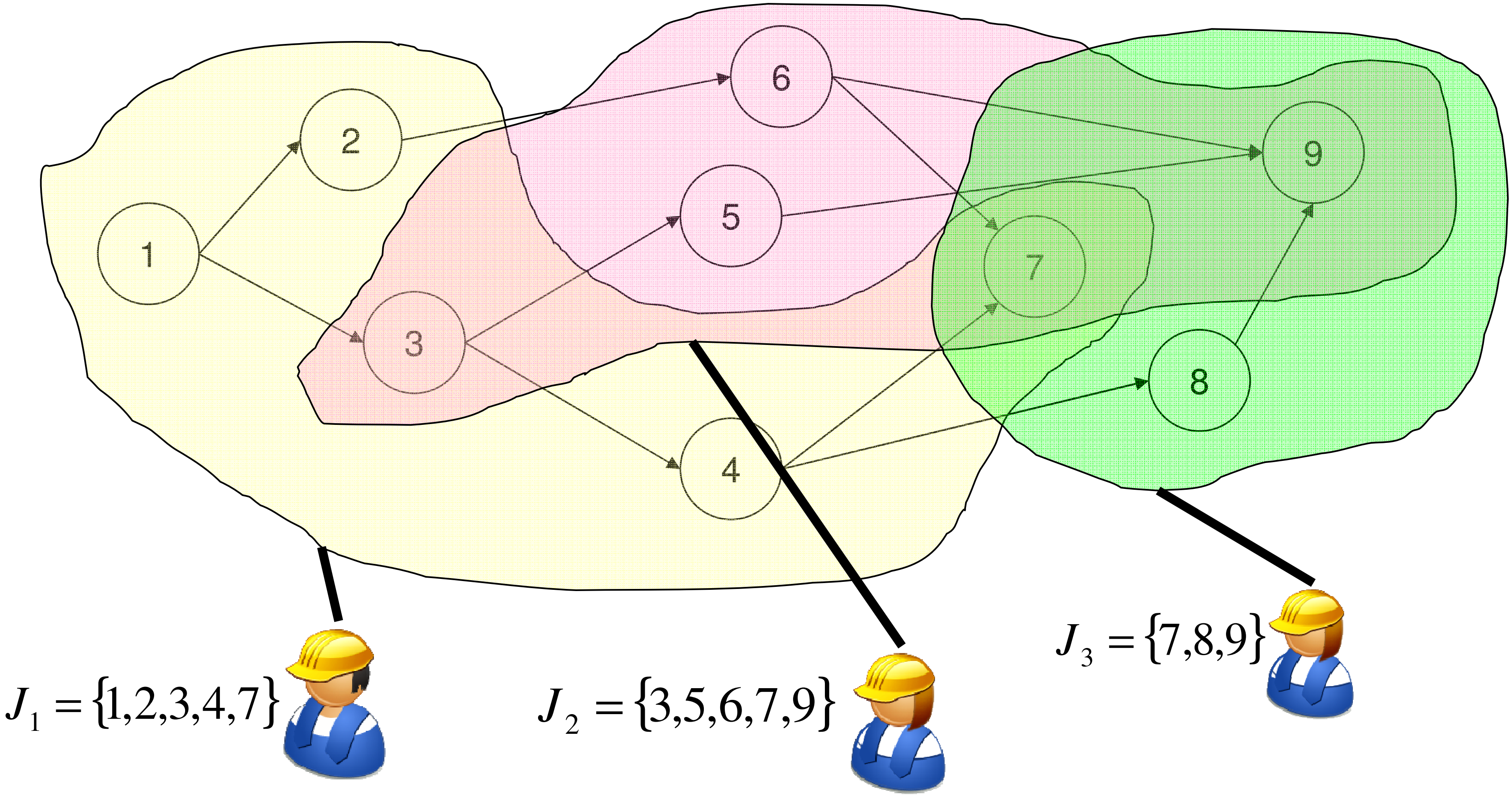
Given

H = set of all workers

J_k = set of tasks that can be done by worker *k*

W_k = hourly wage for worker *k*

Project Management



Assume task durations and "crashing capability" does not depend on "skill" of worker.

Project Management

Seemed like we need the following variable:

y_{ik} = 1, iff task *i* is assigned to worker *k*, for all $i \in J_k$

Model is:

$$\min \quad \sum_{i \in S} C_i x_i + \sum_{k \in H} \sum_{i \in J_k} (D_i - x_i) W_k y_{ik}$$

Non-linear!!

$$s.t.$$

$$t_i + D_i - x_i \le T, \quad \forall i \in S$$

$$t_i \ge t_j + D_j - x_j, \quad \forall i \in S,\ j \in P_i$$

$$x_i \le D_i, \quad \forall i \in S$$

$$\sum_{\substack{k \in H \\ s.t.\, i \in J_k}} y_{ik} = 1, \quad \forall i \in S$$

$$t_i, x_i \ge 0, \quad \forall i \in S$$

$$y_{ik} \in \{0,1\}, \quad \forall k \in H, i \in J_k$$

Project Management

$$\min \quad \ldots + \sum_{k \in H} \sum_{i \in J_k} W_k \; ?$$

Introduce the following variable:

- l_{ik} = duration of task *i* performed by worker *k*

$$l_{ik} \geq D_i - x_i - M\left(1 - y_{ik}\right), \quad \forall k \in H, i \in J_k$$

Project Management

$$\min \quad \sum_{i \in S} C_i x_i + \sum_{k \in H} \sum_{i \in J_k} W_k l_{ik}$$

$$s.t.$$

$$t_i + D_i - x_i \le T, \quad \forall i \in S$$

$$t_i \ge t_j + D_j - x_j, \quad \forall i \in S,\ j \in P_i$$

$$x_i \le D_i, \quad \forall i \in S$$

$$\sum_{\substack{k \in H \\ s.t.\, i \in J_k}} y_{ik} = 1, \quad \forall i \in S$$

$$l_{ik} \ge D_i - x_i - M\left(1 - y_{ik}\right), \quad \forall k \in H, i \in J_k$$

$$t_i, x_i \ge 0, \quad \forall i \in S$$

$$y_{ik} \in \{0,1\}, l_{ik} \ge 0, \quad \forall k \in H, i \in J_k$$

Project Management

But if a set of tasks is assigned to a worker, can the worker carry out the task simultaneously?

Most likely not!!

So for each worker, we need to “sequence” the set of assigned tasks, to ensure each worker performs one task at a time.

Variables:

$z_{ijk} = 1$, iff task *i* precedes task *j* for worker *k*.

Project Management

Now, start time of a task is dependent on

- its **usual** preceding tasks
- its **worker-implied** preceding task

How do we model task start times?

$$t_i \geq t_j + D_j - x_j, \quad \forall i \in S,\ j \in P_i$$

$$t_i \geq t_j + D_j - x_j - M\left(1 - \sum_{\substack{k \in H \\ s.t.\ \{i,j\} \subseteq J_k}} z_{jik}\right),$$

$$\forall i, j \in S\ s.t.\ \left(\exists k \in H\ s.t.\ \{i, j\} \subseteq J_k\right)$$

$$\min \quad \sum_{i\in S} C_i x_i + \sum_{k\in H}\sum_{i\in J_k} W_k l_{ik}$$

$$s.t.$$

$$t_i + D_i - x_i \le T, \quad \forall i \in S$$

$$t_i \ge t_j + D_j - x_j, \quad \forall i \in S,\ j \in P_i$$

$$t_i \ge t_j + D_j - x_j - M\left(1 - \sum_{\substack{k\in H \\ s.t.\{i,j\}\in J_k}} z_{jik}\right), \quad \forall i, j \in S\ s.t. \exists k \in H\ s.t. \{i, j\} \subseteq J_k$$

$$x_i \le D_i, \quad \forall i \in S$$

$$\sum_{\substack{k\in H \\ s.t.\, i\in J_k}} y_{ik} = 1, \quad \forall i \in S$$

$$\sum_{j\in J_k \cup \{O\}} z_{jik} = y_{ik}, \quad \forall k \in H, i \in J_k$$

$$\sum_{i\in J_k} z_{Oik} = \sum_{i\in J_k} y_{ik}, \quad \forall k \in H$$

$$l_{ik} \ge D_i - x_i - M(1 - y_{ik}), \quad \forall k \in H, i \in J_k$$

$$t_i, x_i \ge 0, \quad \forall i \in S$$

$$y_{ik} \in \{0,1\}, l_{ik} \ge 0, \quad \forall k \in H, i \in J_k$$

$$z_{ijk} \in \{0,1\}, \quad \forall k \in H, \{i, j\} \subseteq J_k$$

Employee shift covering

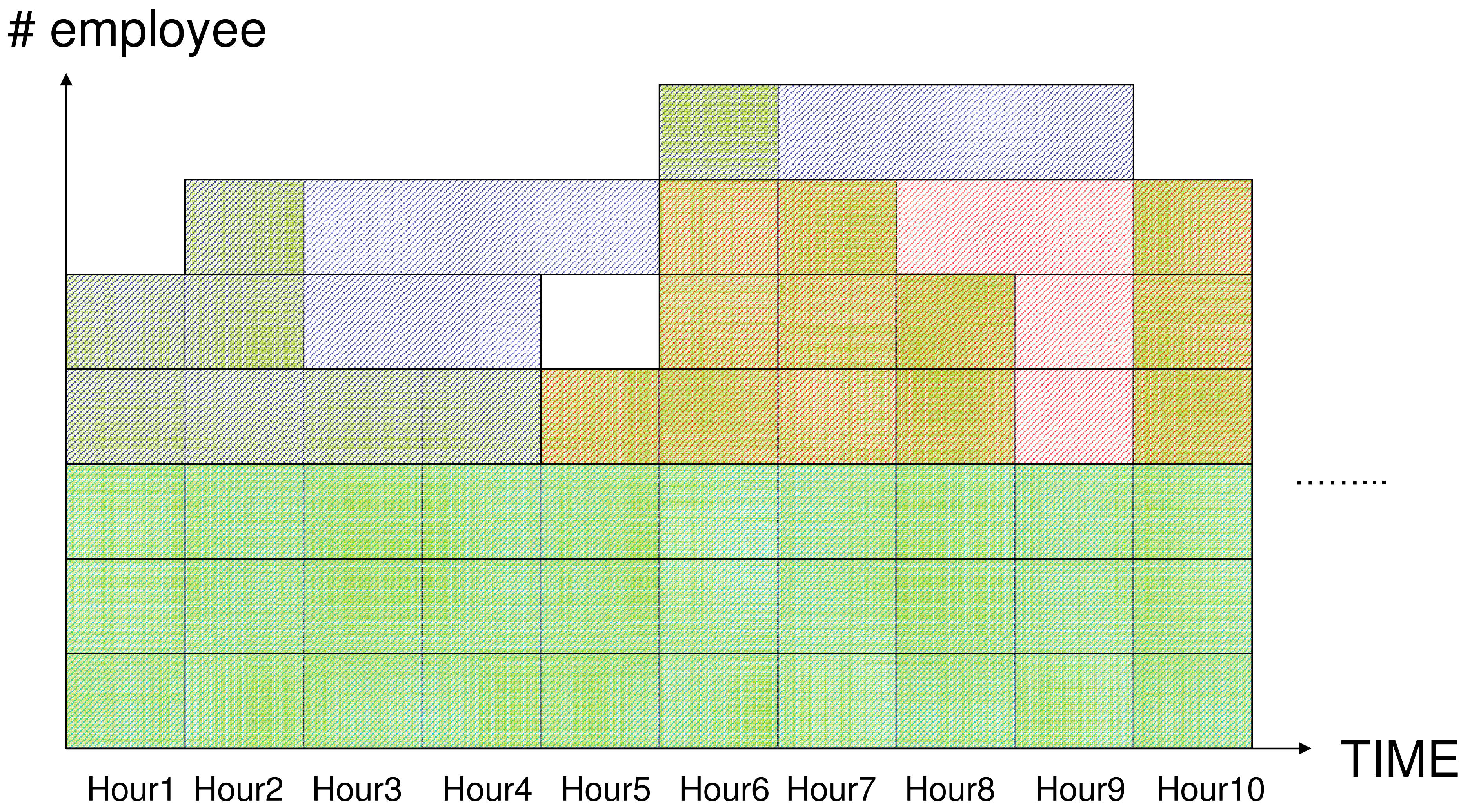
Given

- S = set of all shift types (Casual-4hr,Casual-5hr, Fulltime-8hr, Fulltime-9hr etc.)
- {1,…,T} is the set of all hours
- D_i = employee demand for hour *i*
- W = the total number of employees
- H = total work limit
- C_i = cost of shift type $i \in S$
- L_i = length of shift type $i \in S$

Objective:

- Determine the number of each shift type to use to minimal overall cost, such that hourly demand is met.

Employee shift covering



Employee shift covering

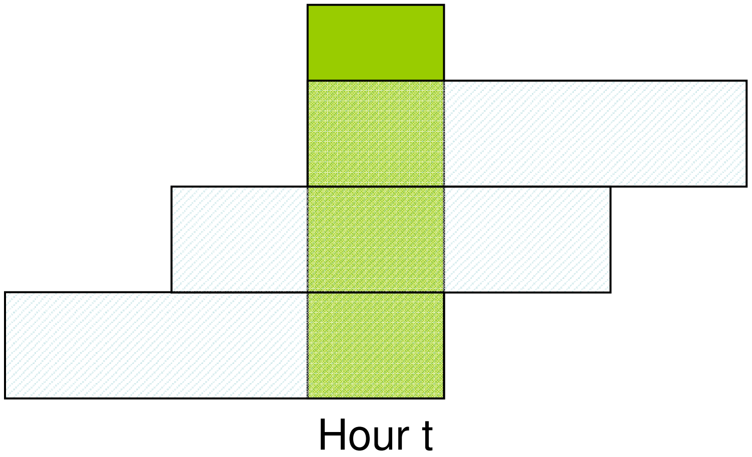
Variables:

- x_{tq} = # shift type q starting at hour t
- n_t = # employees in hour t

Relationship between n_t and x_{tq}:

- Assume shift pattern not repeated over planning horizon

$$n_t = \sum_{q \in S} \sum_{r=t-L_q+1}^{t} x_{rq}$$



Employee shift covering

$$\min \quad \sum_{q\in S}\sum_{t=1}^{T} C_t x_{tq}$$

$$s.t.$$

$$n_t = \sum_{q\in S}\sum_{r=t-L_q+1}^{t} x_{rq}, \quad \forall t = 1,\ldots,T$$

$$D_t \le n_t \le W, \quad \forall t = 1,\ldots,T$$

$$\sum_{q\in S}\sum_{t=1}^{T} L_q x_{tq} \le H$$

← What is the use of this constraint?

Employee shift covering

HOMEWORK:

What if shift pattern has to be repeated over a rolling time horizon?