620-362
Applied Operations Research

Network Models

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Some contents of this presentation are adapted from year 2005 course notes for 620-362 Applied Operations Research, Department of Mathematics and Statistics, The University of Melbourne (compiled by Prof Natashia Boland and Dr Renata Sotirov)

Network Models

- very special mathematical programs
- extremely fast and efficient techniques available
- enormous problems can be solved
- special properties e.g. naturally integer
  - total unimodularity
- intuitive, natural paradigm for thinking about real problems
- many, many real problems have network structure

Network Models

References:

- R. Ahuja, T. Magnanti & J. Orlin
  "Network Flows: Theory, Algorithms and Applications"

- Winston Chapters 7 & 8

We will look at

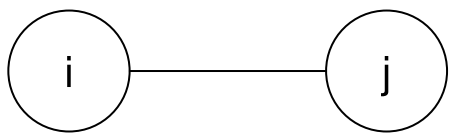
- spanning trees
- shortest paths
- network flows
- multicommodity flows

Terms and Notation

Graph, Digraph

N = set of nodes or vertices

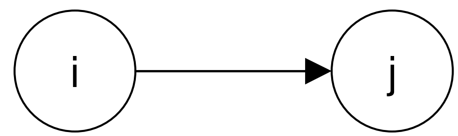
A = set of edges or arcs or links



$\{i, j\}$

undirected edge

unordered node pair



(i, j)

directed edge

ordered node pair

Graph = (N,A)

Digraph = (N,A), A consists of directed edge

Terms and Notation

Network

- Graph or digraph + extra information
- e.g. cost of an edge, c_{ij}, for edge (i,j)

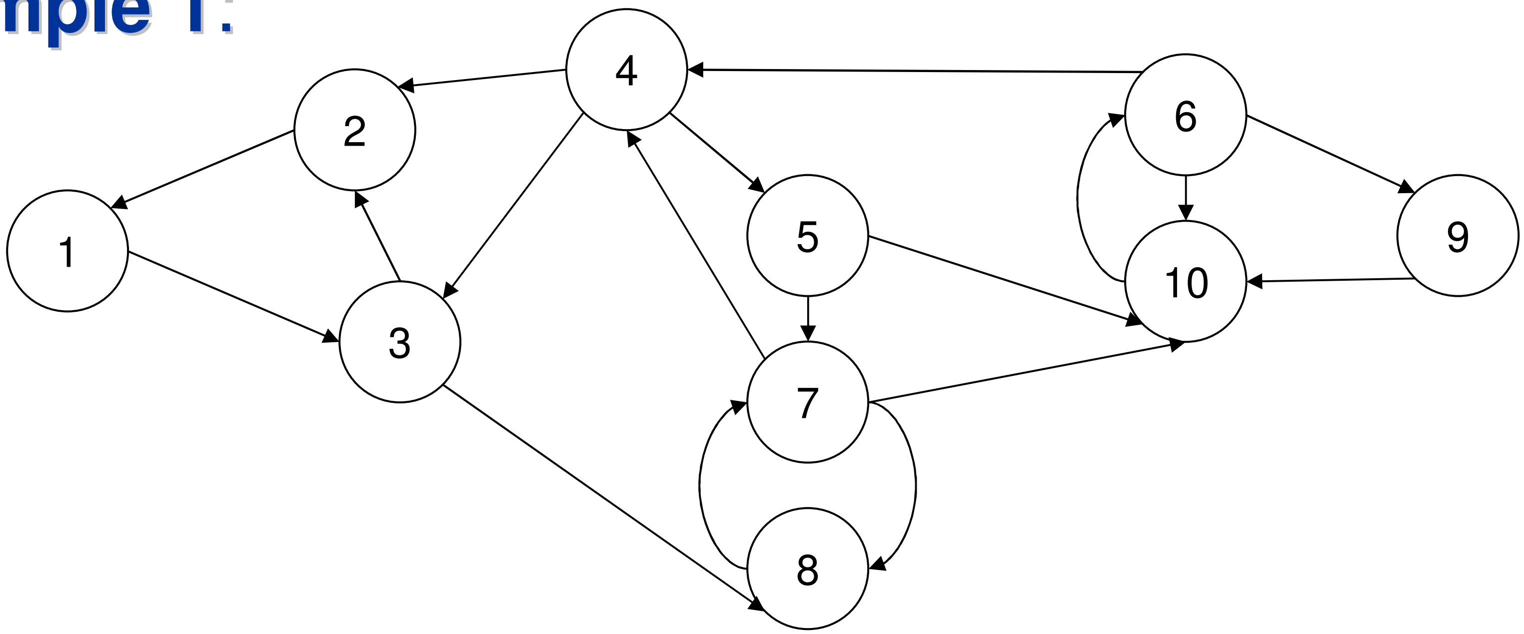
Path from node $s \in N$ to node $t \in N$

- s is referred to as the "source node"
- t is referred to as the "terminal node", "sink" or "destination"

A path is a sequence of nodes $(s = i_0, i_1, \ldots, i_n = t)$, $\{i_0, i_1, \ldots, i_n\} \subseteq N$, $(i_{j-1}, i_j) \in A$ for all $j=1,\ldots,n$

Terms and Notation

Example 1:



N = {1,2,…,10}
A = {(1,3), (2,1), (3,2), (3,8), (4,2), (4,3), (4,5), …}

A path from 1 to 9 is
(1,3,8,7,4,5,10,6,9)

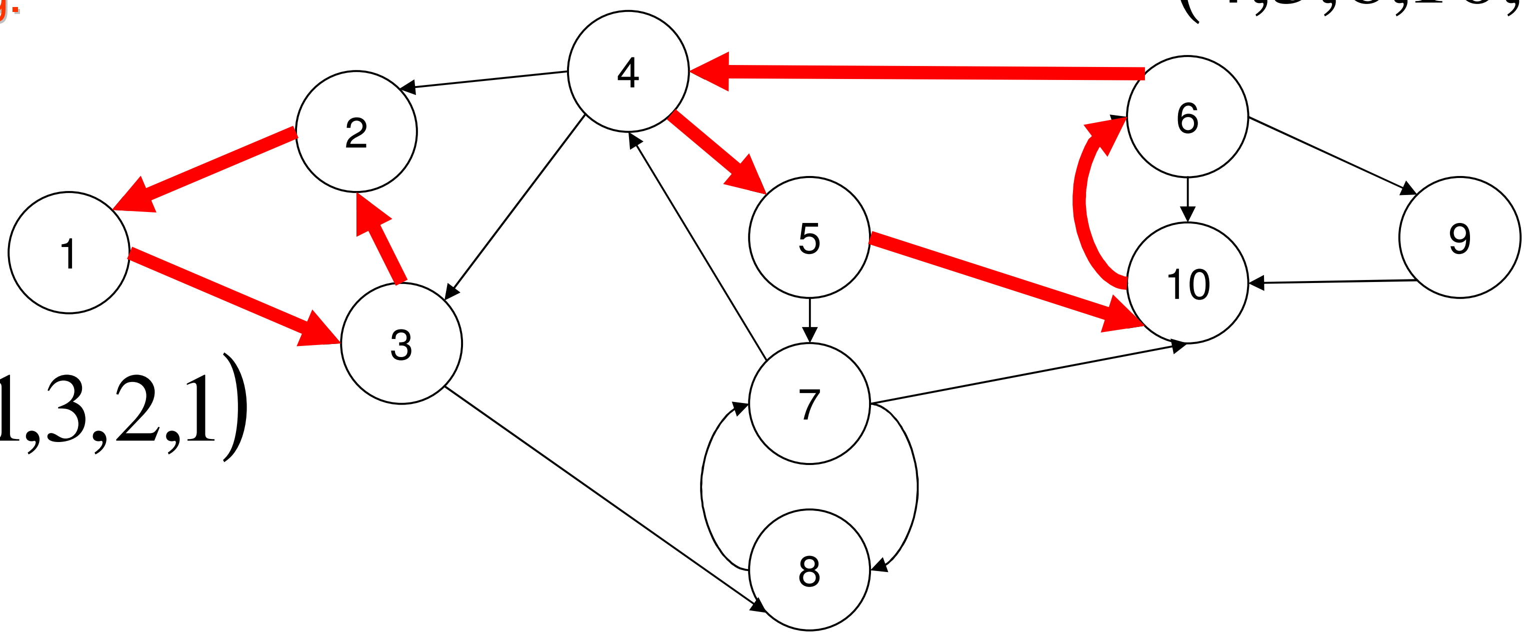
Another path from 1 to 9 is
(1,3,8,7,10,6,9)

Terms and Notation

Simple cycle

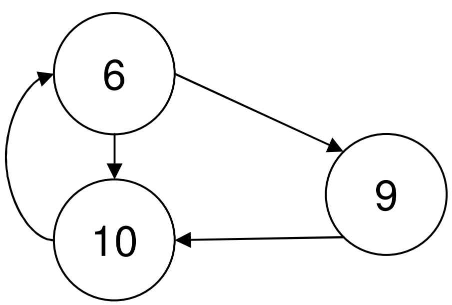
a path from $s \in N$ to itself in which no node is repeated
e.g.

$(4,5,6,10,6,9)$



$(1,3,2,1)$

except s, which occurs only at the beginning and end.

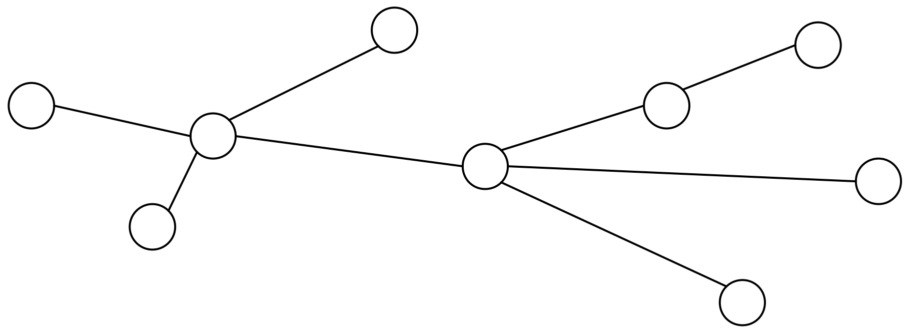


$(9,10,6,10,6,9)$ is a cycle

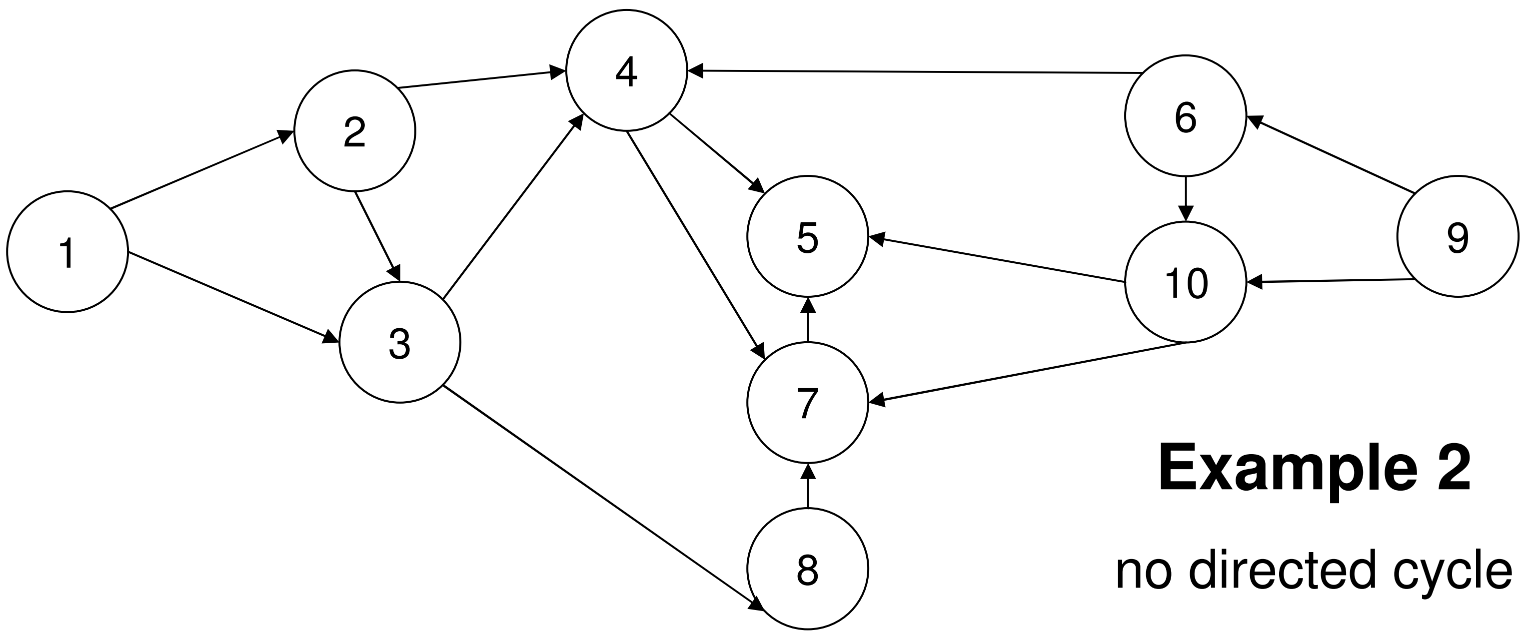
Terms and Notation

Acyclic graph

does not contain any cycles



no cycles



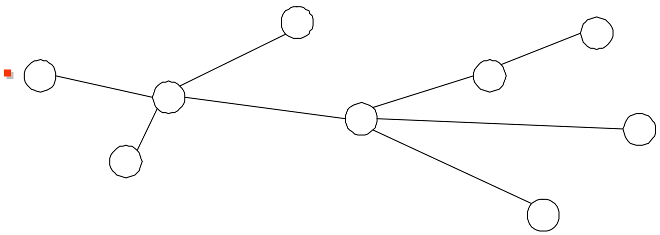
Example 2

no directed cycle

Terms and Notation

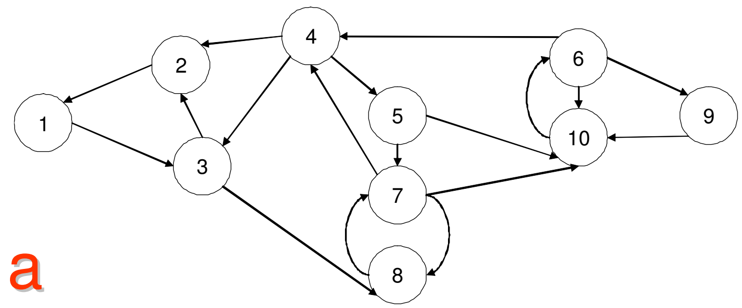
Tree

undirected, acyclic, connected graph.

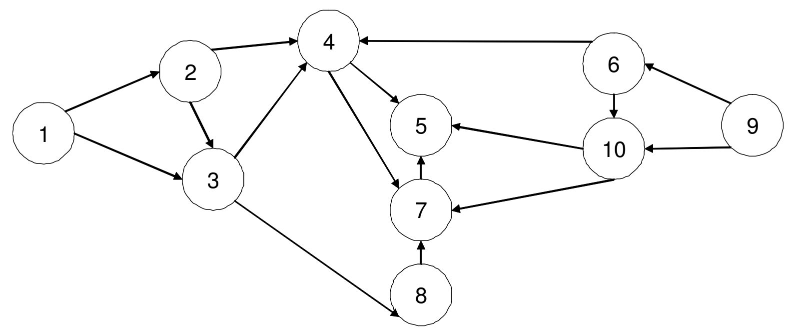


Predecessor of node i∈N

any node j∈N such that there is a path from j to i in the graph



e.g. from **Example 1**, 1 is a predecessor of 7 since (1,3,8,7) is a path



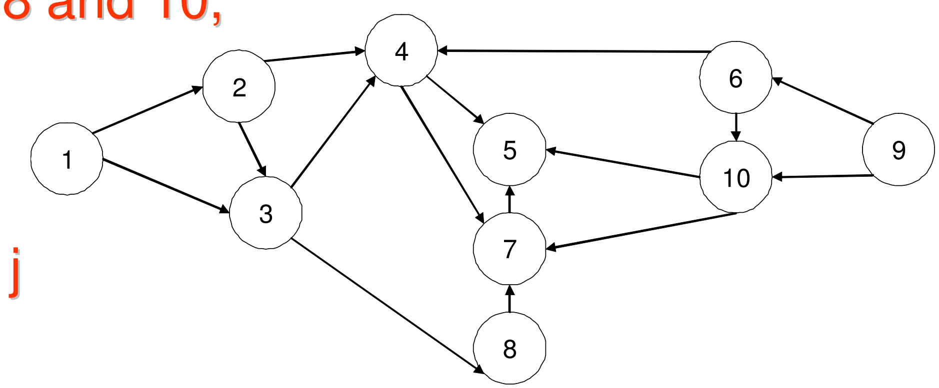
e.g. from **Example 2**, 1 is a predecessor of 7 but 5 is not

Terms and Notation

Immediate predecessor of node *i*

is a node *j* where $(j,i) \in A$; the set of these nodes is denoted P(i)

e.g. from **Example 2**, immediate predecessors of 7 are nodes 4, 8 and 10, so P(7) = {4,8,10}



Successor of $i \in N$

any node $j \in N$ with a path from i to j

Immediate successor of node i

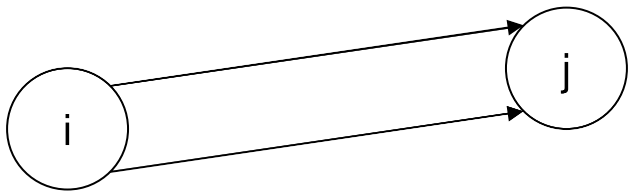
is node j, where $(i,j) \in A$, the set of these nodes is denoted S(i)

e.g. from **Example 2**, immediate successors of 3 are nodes 4 and 8, so S(3) = {4,8}

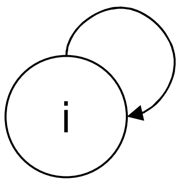
General Assumptions

For all nodes $i,j \in N$:

1. there is at most one arc $(i,j) \in A$, i.e. there are no "parallel" edges.



2. there are no arcs (i,i), i.e. there are no "loops".



$(A \subseteq N \times N, \quad (i,i) \notin A, \quad \text{forall } i \in N)$

Properties

For a digraph,

$$|A| \leq |N|(|N|-1) \approx |N|^2$$

For an undirected graph,

$$|A| \leq \frac{1}{2}|N|(|N|-1) \approx \frac{1}{2}|N|^2$$

In general, we expect $|A|$ is $O(|N|^2)$

Trees in Undirected Networks

Definition

Given a graph with nodes 1,…,N and arcs $(i,j) \in A$:

1. The graph is connected if for any two nodes, there is at least one path connection these nodes; otherwise the graph is disconnected.
2. The graph is a tree if it is undirected, connected and acylic.
3. A subgraph is a spanning tree if it is a tree containing every vertex of the graph.

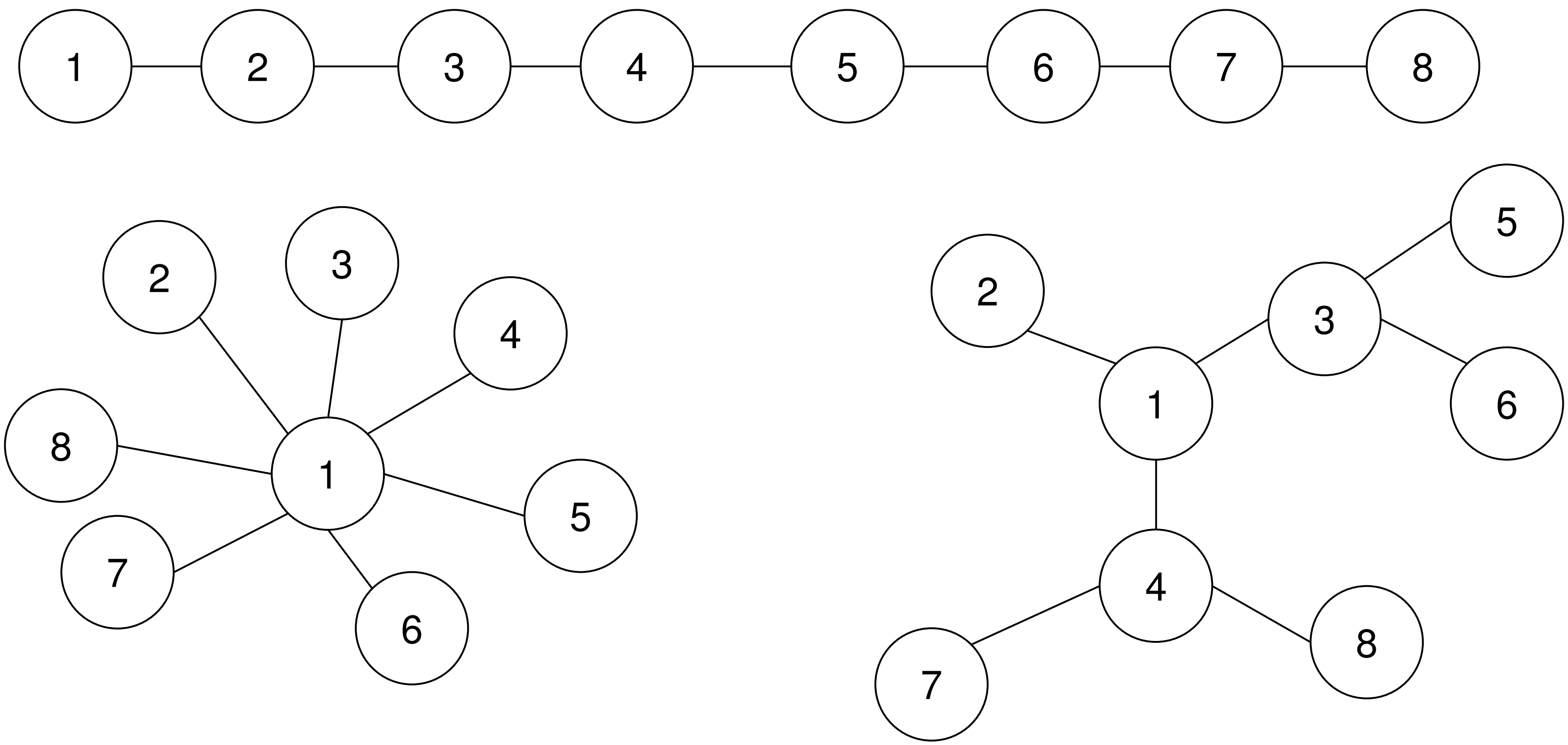
Given an undirected network, with arc lengths a_{ij} for $(i,j) \in A$:

4. A minimal spanning tree or MST is the spanning tree whose cost or weight, i.e. the sum of all the arcs lengths of the tree, is the least cost amongst all spanning trees.

Spanning Tree

Note: a spanning tree in a graph of N nodes has N nodes and (N-1) arcs.

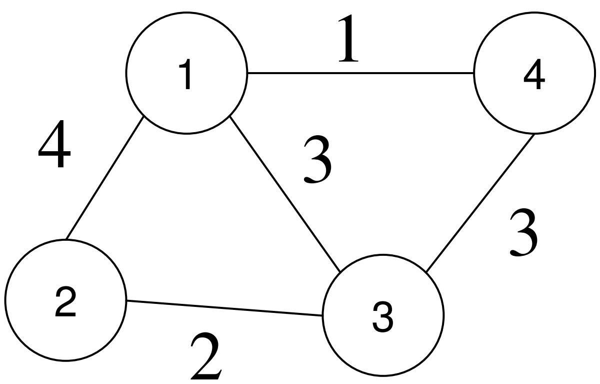
Some spanning trees on 8 nodes…all have 7 arcs!



Spanning Tree

Exercise

Write down all the spanning trees of



$\{\{1,2\},\{1,4\},\{1,3\}\},\quad Cost = 4+1+3=8$

$\{\{1,3\},\{2,3\},\{3,4\}\},\quad Cost = 3+2+3=8$

$\{\{1,4\},\{3,4\},\{2,3\}\},\quad Cost = 1+3+2=6$

$\{\{1,2\},\{2,3\},\{3,4\}\},\quad Cost = 4+2+3=9$

$\{\{1,4\},\{1,2\},\{2,3\}\},\quad Cost = 1+4+2=7$

$\{\{1,3\},\{2,3\},\{1,4\}\},\quad Cost = 3+2+1=6$

$\{\{1,2\},\{1,3\},\{3,4\}\},\quad Cost = 4+3+3=10$

$\{\{1,2\},\{1,4\},\{3,4\}\},\quad Cost = 4+1+3=8$

What is an MST?

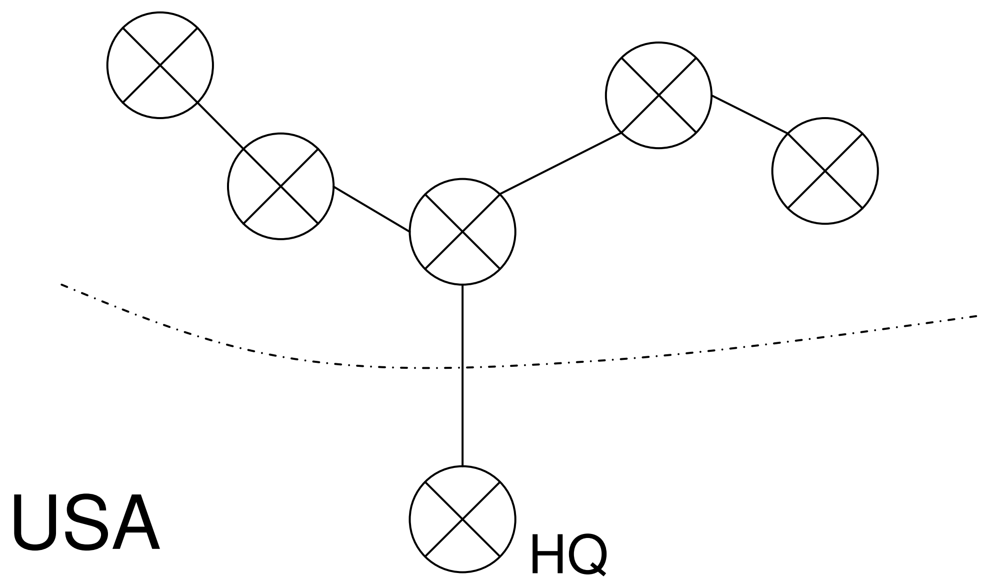
What is its cost?

Spanning Tree

Applications (Two examples)

Geographic Network Design

Cable TV in Canada where cable network originates in northern USA: Form an MST connecting cable HQ with Canadian towns/cities.



Spanning Tree

Data compression

Store vectors of similar data, e.g. sequences of amino acids in genetics, using similarities to reduce volume of data required.

For each pair of vectors V_i, V_j (i=1,…,N), let

a_{ij} = # if components in v_i different from the corresponding components of v_j.

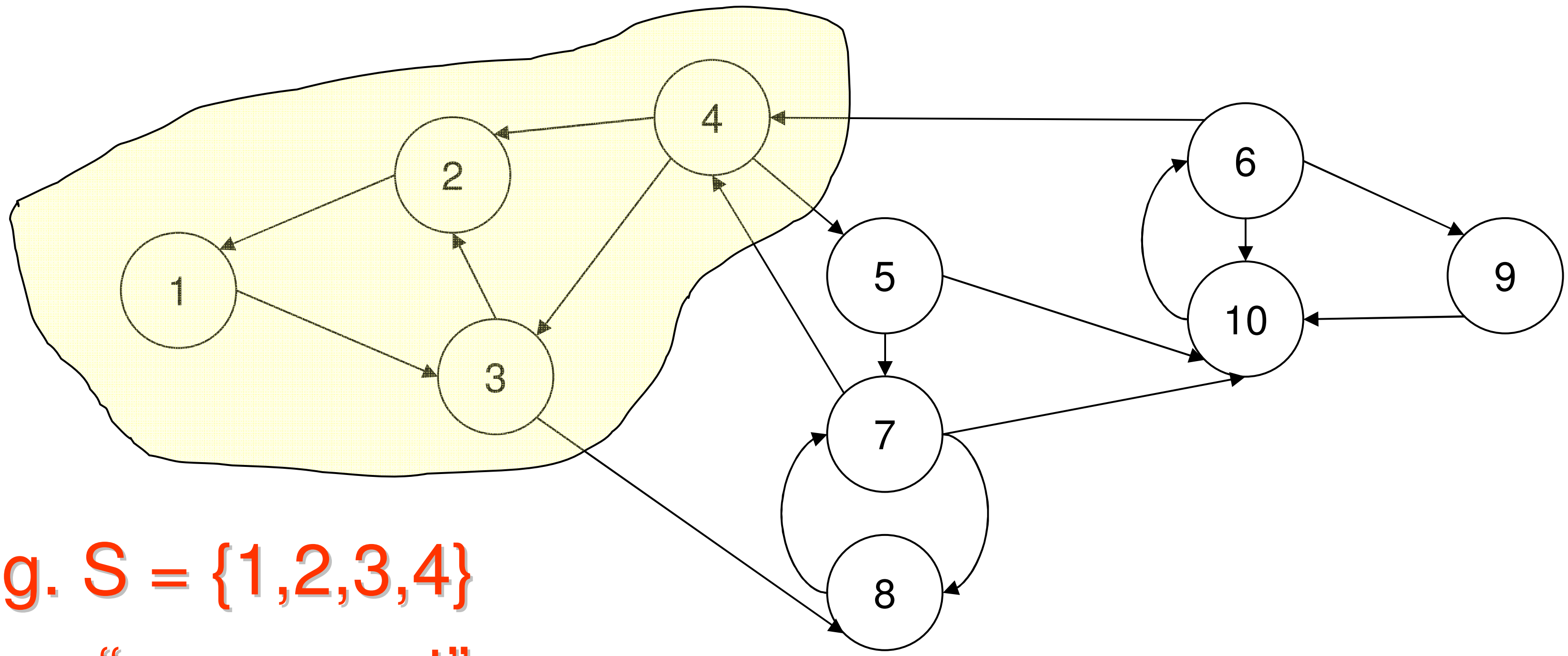
Find an MST in the network of nodes $V_1,\ldots,V_N$ where there is an arc of cost a_{ij} between each V_i and V_j ($i \neq j$).

Choose V_1 as the "reference" vector. For each immediate successor V_j of V_1 in the MST, store only the differences between V_j and V_1. Continue this process from V_j.

Spanning Tree

Cut

a subset of nodes S⊆V



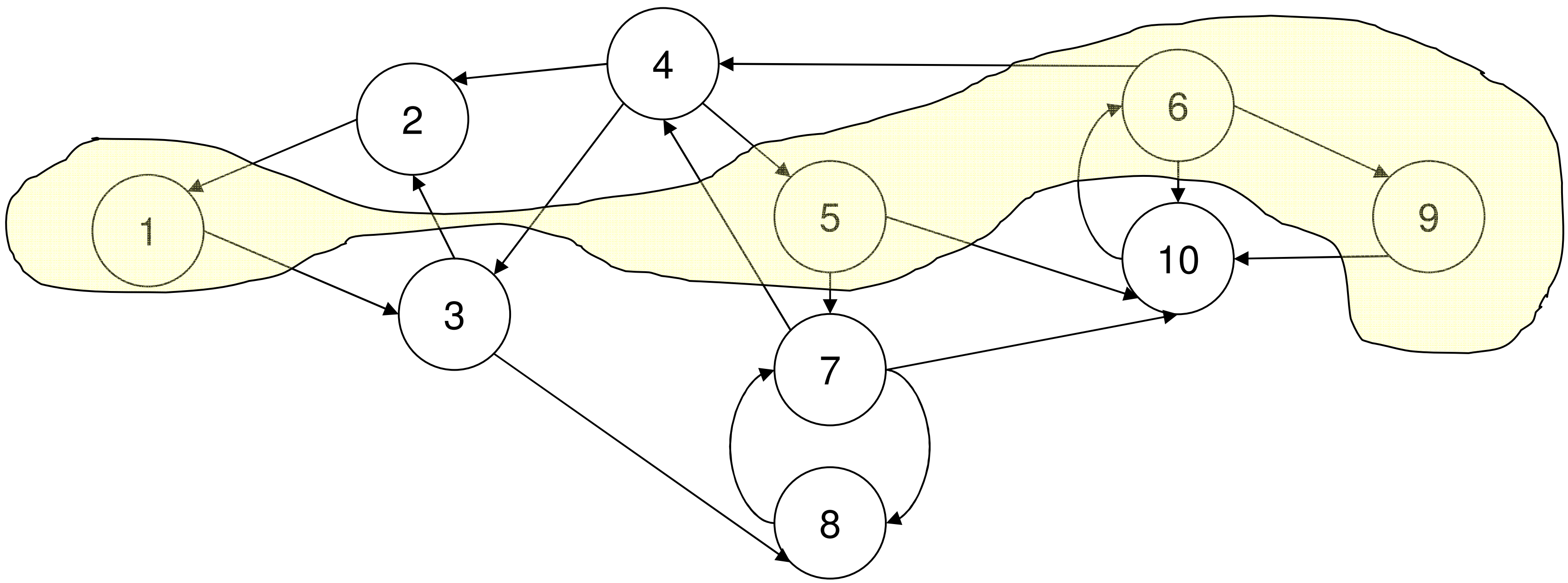
e.g. S = {1,2,3,4}

arcs "across cut" are:

arcs "out of cut": (4,5), (3,8)

arcs "into cut": (6,4), (7,4)

Spanning Tree



e.g. S = {1,5,6,9}

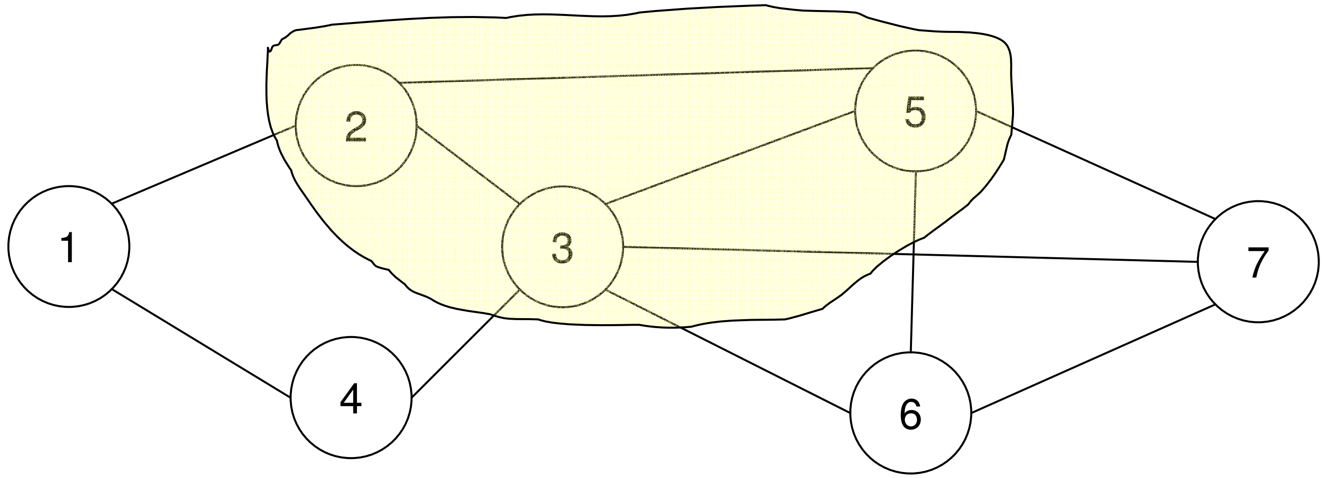
arcs "out of cut":

(1,3), (5,10), (5,7), (6,10), (6,4), (9,10)

arcs "into cut":

(2,1), (4,5), (10,6)

Spanning Tree



e.g. S = {2,3,5}

edges across cut:

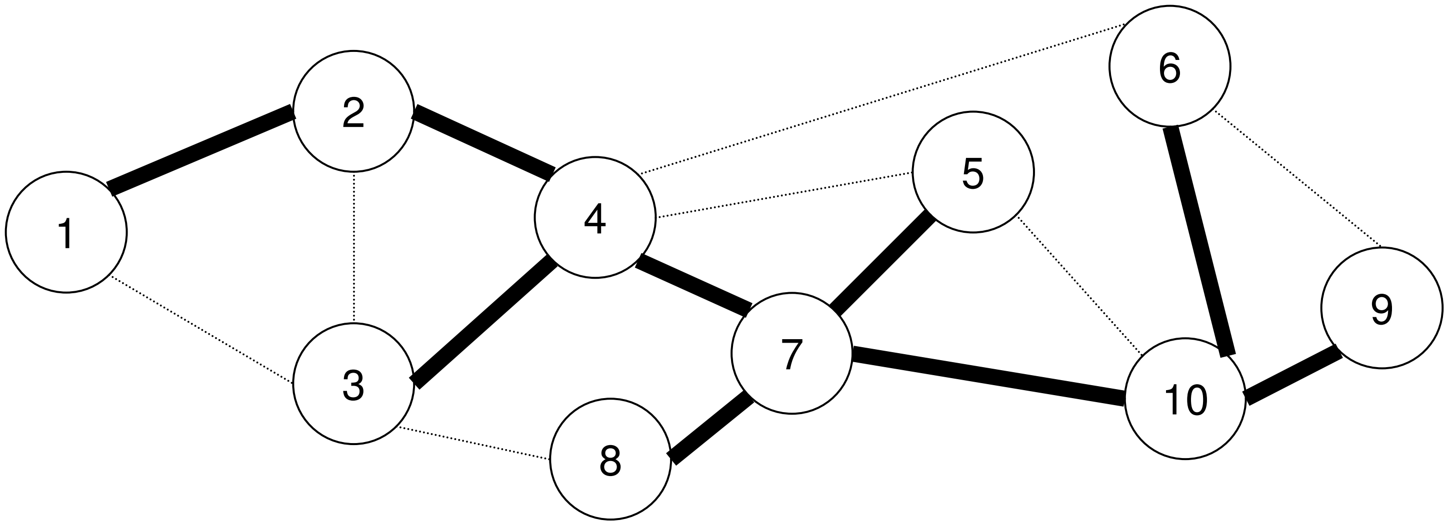
{1,2}, {3,4}, {3,6}, {3,7}, {5,6}, {5,7}

e.g. S = {1,4,6,7} = V\{2,3,5}

edges across cut are the same!!

Spanning Tree Properties

1A. Any edge in a spanning tree induces a cut in the graph: remove the edge and the spanning tree breaks into 2 disconnected components – nodes of one component form the cut.

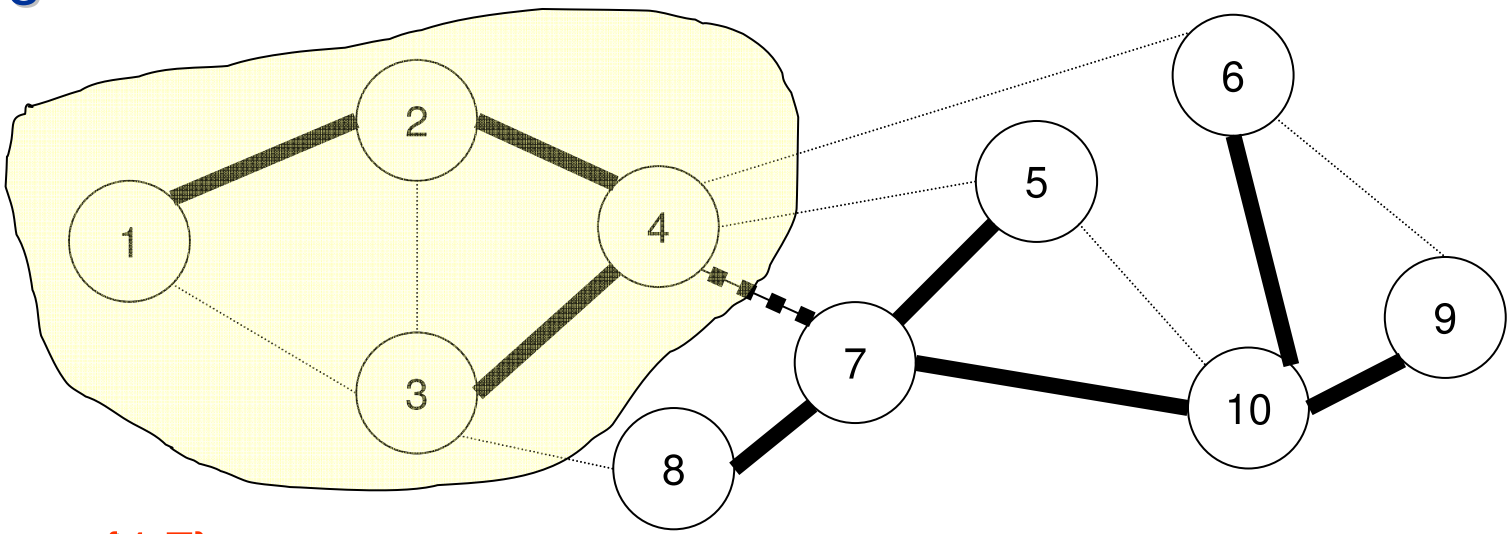


Consider spanning tree T = {{1,2}, {2,4}, {3,4}, {4,7}, {5,7}, {7,10}, {6,10}, {7,8}, {9,10}}

Edge {4,7} removed: cut?
Edge {5,7} removed: cut?
Edge {7,10} removed: cut?

Spanning Tree Properties

1B. If edge {i,j} is in spanning tree T, {i,j} induces a cut S⊆V, and {i',j'} is an edge across the cut, then T\{{i,j}} U {{i',j'}} is also a spanning tree.



e.g. remove {4,7}

edges across cut are

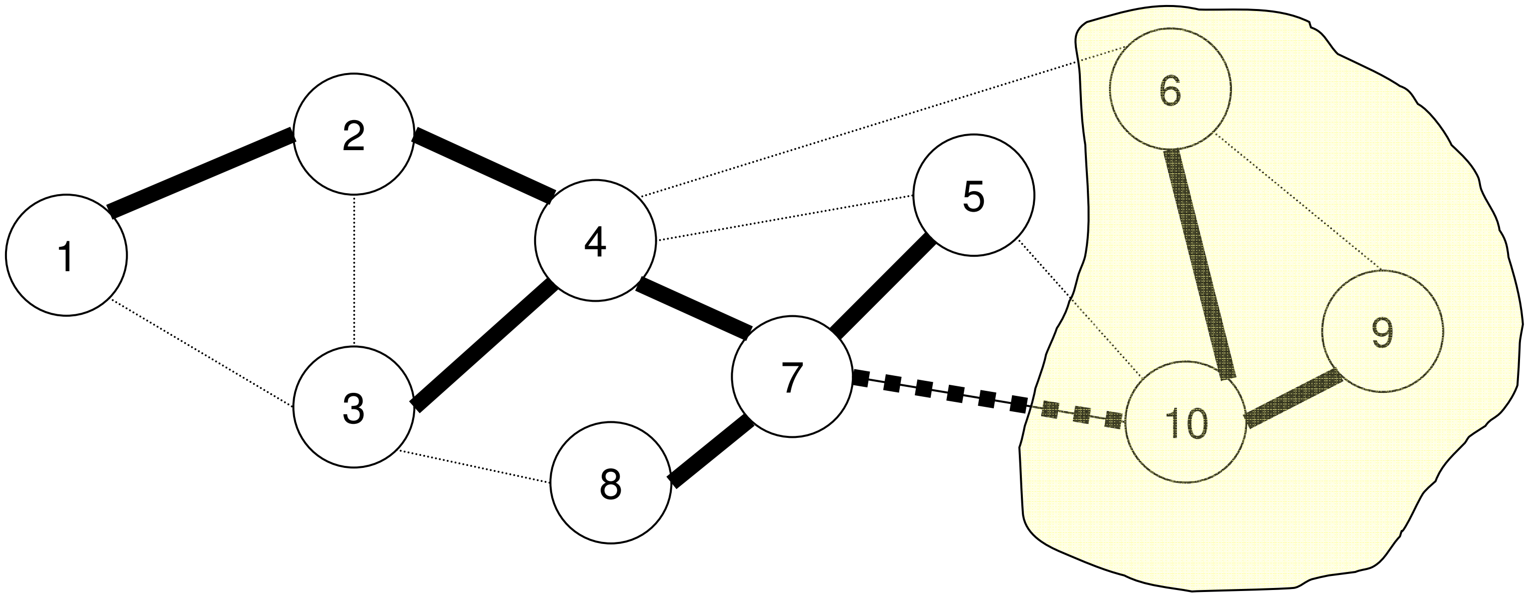
{3,8}, {4,7}, {4,5}, {4,6}

T\{4,7} U {3,8} is a spanning tree

T\{4,7} U {4,5} is a spanning tree

T\{4,7} U {4,6} is a spanning tree

Spanning Tree Properties



e.g. remove {7,10}

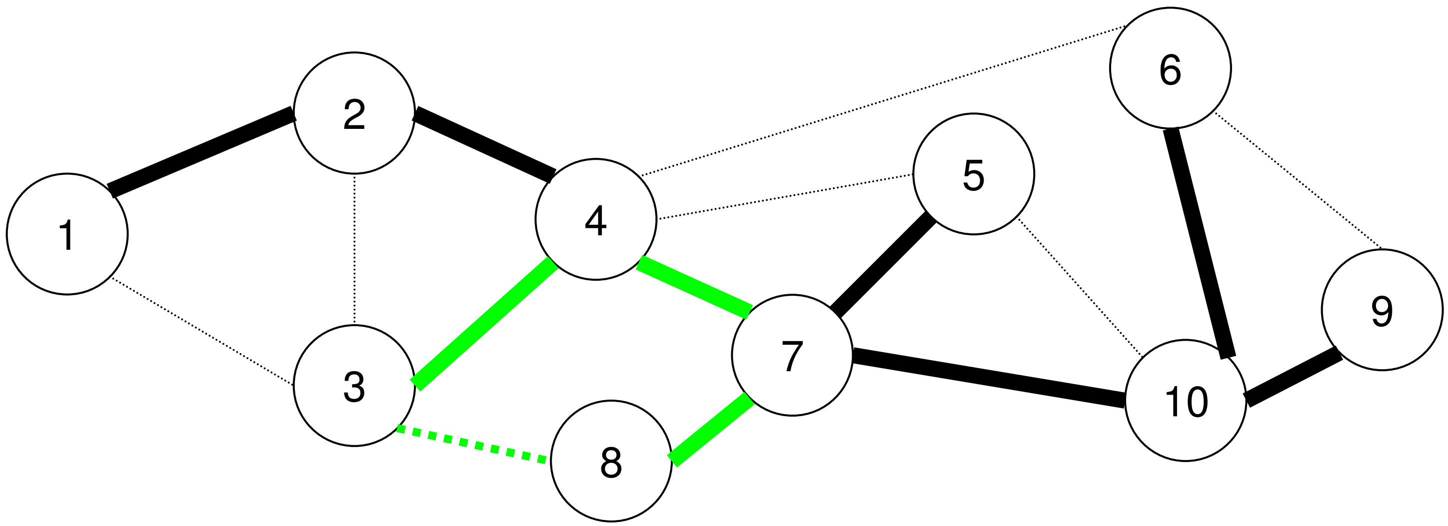
edges across cut are {7,10}, {5,10}, {4,6}

T\{7,10} U {5,10} is a spanning tree

T\{7,10} U {4,6} is a spanning tree

Spanning Tree Properties

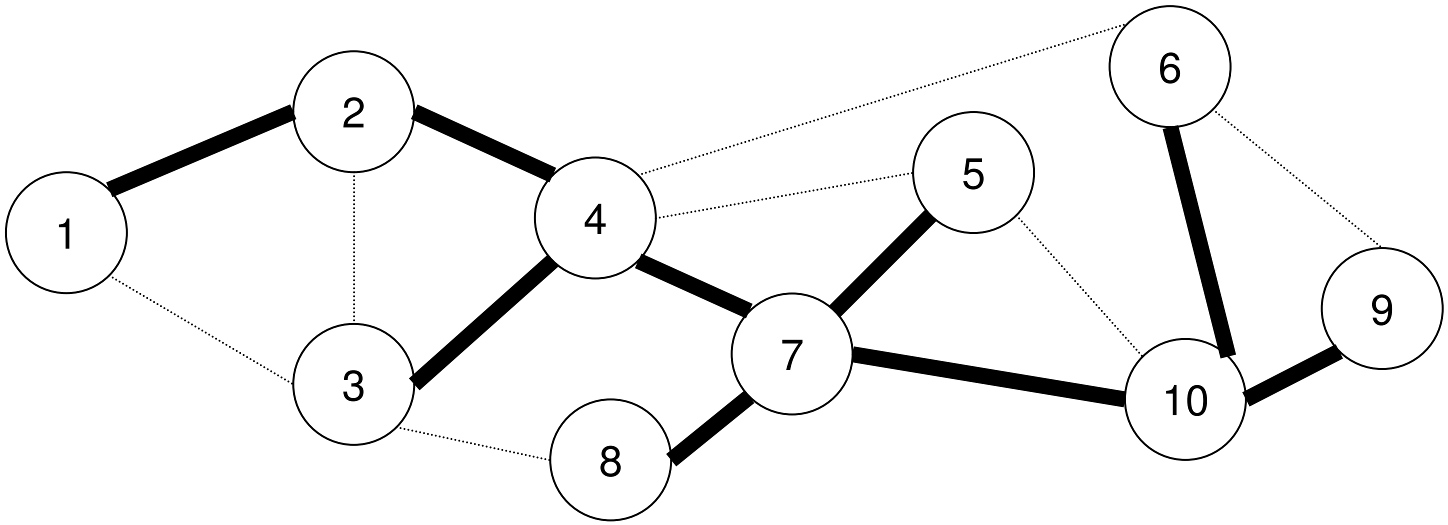
2A. Any edge not in a spanning tree induces a unique path in the spanning tree, between the terminal vertices of the edge.



e.g. for {3,8}, path is 3-4-7-8
for {4,6}, path is 4-7-10-6
for {4,5}, path is 4-7-5

Spanning Tree Properties

2B. If {i,j} is not in spanning tree T, and {i',j'} is some edge on unique path in T between i and j, then T\{i',j'} U {i,j} is also a spanning tree.



e.g.

T\{3,4} U {3,8} is a spanning tree

T\{4,7} U {3,8} is a spanning tree

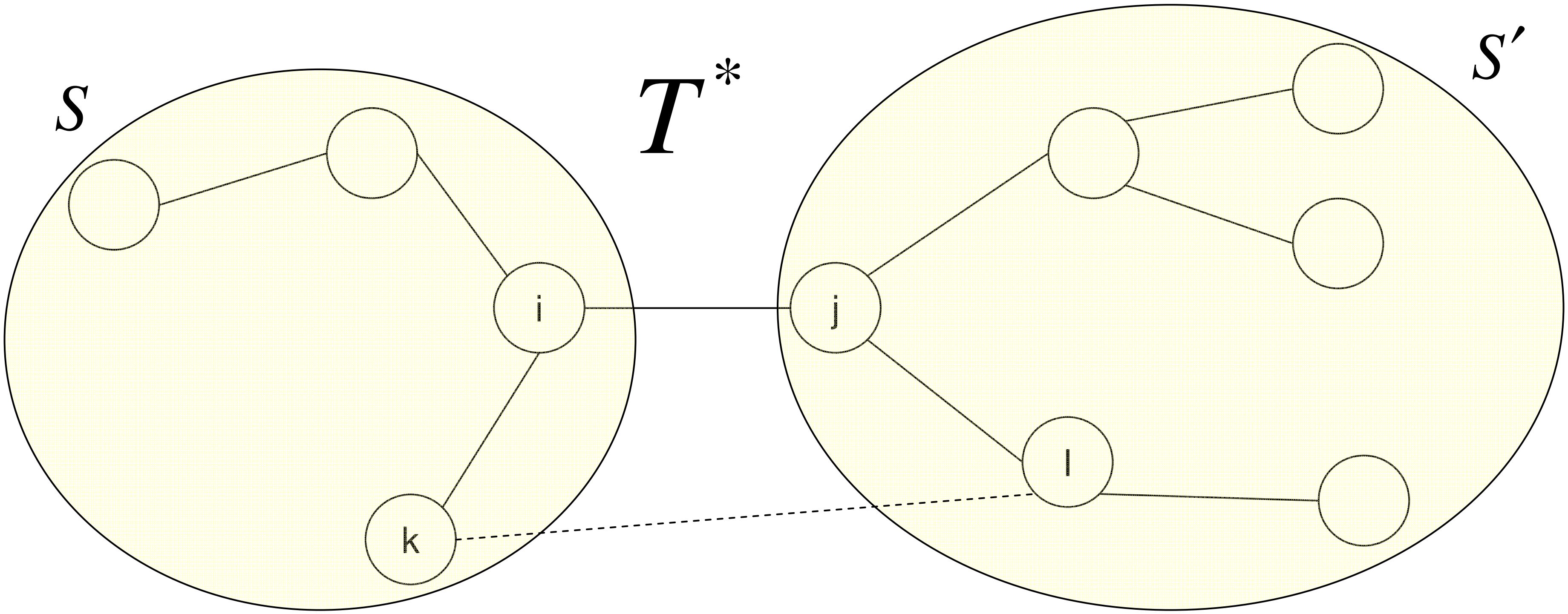
T\{7,8} U {3,8} is a spanning tree

…

Cut Optimality Condition

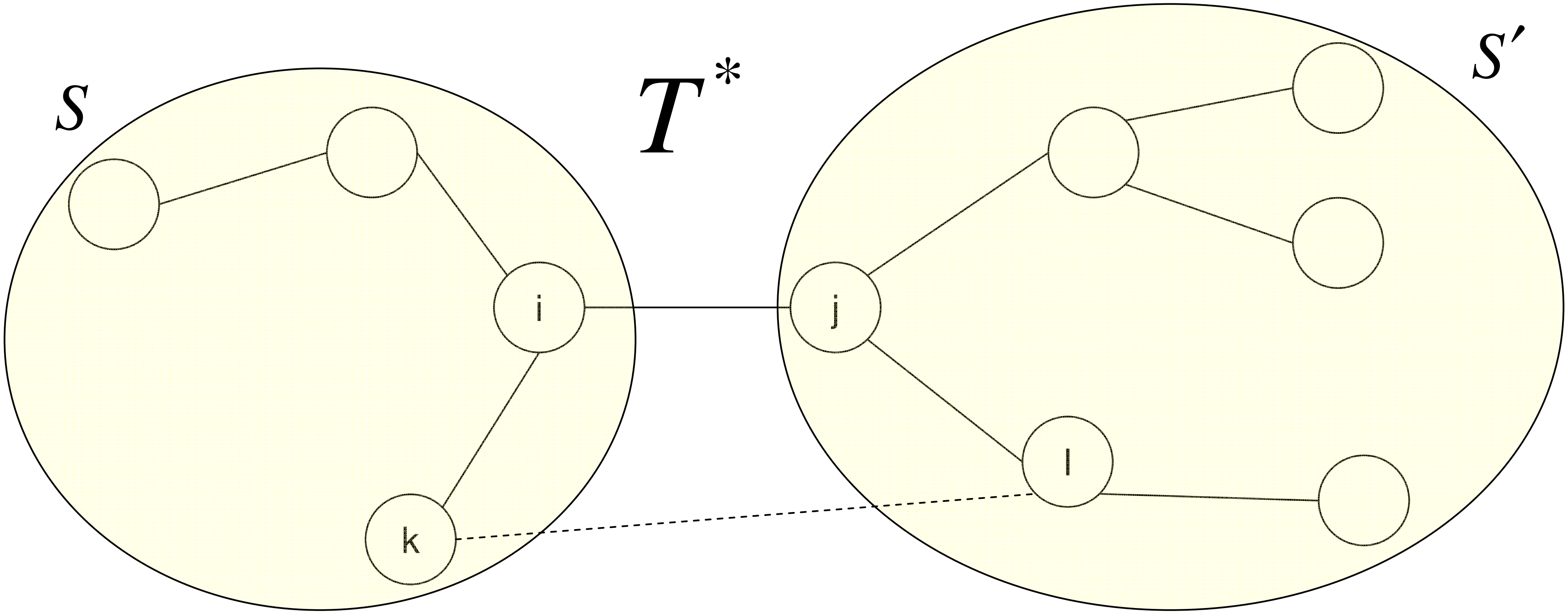
A spanning tree T^* is an MST iff for every arc $\{i,j\} \in T^*$,

$c_{ij} \leq c_{kl}$ for all $\{k,l\}$ contained in the cut formed by deleting $\{i,j\}$ from T^*



$\Rightarrow$ If $c_{ij} > c_{kl}$ then new tree $(T^* \cup \{k,l\}) \setminus \{i,j\}$ is a cheaper spanning tree. Therefore, contradicts minimality of T^*.

Cut Optimality Condition



⇐ Suppose T^* satisfies cut condition, T^0 is MST and T^* is different from T^0.
Say $\{i,j\} \in T^* \setminus T^0$.
Consider cut S induced by {i,j} in T^*.
$\exists \{k,l\}$ with $k \in S$, $l \in S'$, $\{k,l\} \in T^0$.
Then $c_{ij} = c_{kl}$ otherwise T^0 could be improved!
So $T^0 := (T^0 \cup \{i,j\}) \setminus \{k,l\}$ is MST.
Repeat until $T^0 = T^*$.

Path Optimality Condition

A spanning tree T^* is a MST with respect to cost *c* iff for all $\{k,l\} \in A \setminus T^*$, every arc $\{i,j\}$ in the path connecting k and l in T^* has $c_{ij} \leq c_{kl}$.

$\Rightarrow$ if $\{i,j\} \in T^*$ on path from k to l and $c_{ij} > c_{kl}$
then $(T^* \cup \{k,l\}) \setminus \{i,j\}$ gives cheaper spanning tree.
Therefore, contradicts minimality.

$\Leftarrow$ Suppose T^* satisfies path condition. Let $\{i,j\} \in T^*$ and consider cut S induced by $\{i,j\}$ in T^*. If arc $\{k,l\}$ crosses cut, then the (only) path from k to l in T^* must use $\{i,j\}$. So $\{i,j\}$ is on path connecting k and l, thus $c_{ij} \leq c_{kl} \Rightarrow T^*$ satisfies cut condition $\Rightarrow T^*$ is MST.