620-362
Applied Operations Research

Mid-Semester Test Revision

Department of Mathematics and Statistics
The University of Melbourne

Some contents of this presentation are adapted from year 2005 course notes for 620-362 Applied Operations Research, Department of Mathematics and Statistics, The University of Melbourne (compiled by Prof Natashia Boland and Dr Renata Sotirov)

Overview: What to revise?

- Mixed integer program modelling
  - Piecewise linear functions
  - Big-M constraints
  - Other “creative” use of binary/integer variables
  - “Good” and “bad” models

- Branch-and-bound
  - The branch-and-bound procedure.
  - Use graphical method to solve LP.

- The Gomory cutting plane method
  - Proofs
  - Derive Gomory cuts
  - The dual simplex method

- Column generation & Dantzig-Wolfe Reformulation
  - Pricing subproblem
  - D-W reformulation

- Modelling quadratic models
  - Determine covariance matrix
  - Model problem

MIP Modelling

Integer Programming Modelling

- Knapsack Problems (Capital Budgeting)
- Set Covering/Partitioning/Packing (Human resource planning)
- Facility Location Problems
- Lot-sizing Problems
- Logic and Disjunctive Constraints
- Piecewise Linear Functions
- Quadratic Assignment Problems
- Hub Location Problems (Postal Service Planning)
- Vehicle Routing/Travelling Salesperson

Modelling Logic And Disjunctive Constraints

Logical statements, such as implications, can be modelled using binary variables.

Facility location problem: no more than 10 new facilities to be constructed

$$\sum_{i=1}^{m} y_i \leq 10$$

Can put facilities at both sites 1 and 2 or at neither

$$y_1 = y_2$$

Cannot put facilities at both sites 3 and 4

$$y_3 + y_4 \leq 1$$

If put facility at site 5, must also put facility at site 6

$$y_5 \leq y_6$$

Disjunctive Inequalities

Machine maintenance scheduling:

- Schedule time t to perform maintenance on machine
- Machine will be in use between time T_1 and T_2
- Deadline on maintenance is time T

Disjunctive constraint:

$$t \le T_1 \quad \text{or} \quad t \ge T_2$$

$$y = \begin{cases} 1, & t \le T_1 \text{ is enforced} \\ 0, & t \ge T_2 \text{ is enforced} \end{cases}$$

$$t \le T_1 + (T - T_1)(1 - y)$$

$$t \ge T_2 - T_2 y$$

Modelling Logical Relations

Select at most one (packing): $\sum_{j \in S} x_j \leq 1$

Select precisely one (partitioning): $\sum_{j \in S} x_j = 1$

Select at least one (covering): $\sum_{j \in S} x_j \geq 1$

Special conditions:

$x_1 = 1 \Rightarrow x_2 = 1$ $\qquad$ $x_1 - x_2 \leq 0$

$x_1 = 1 \Rightarrow x_2 = 0$ $\qquad$ $x_1 + x_2 \leq 1$

$x_1 = 1 \wedge x_2 = 1 \Rightarrow x_3 = 1$ $\qquad$ $x_1 + x_2 - x_3 \leq 1$

Production Scheduling

$$\min \max_{t=2,\dots,12} |l_t - l_{t-1}|$$

$$s.t. \quad l_t = b_1 x_{1t} + b_2 x_{2t}, \; \forall t = 1,\dots,12$$

$$y_1 = x_{11} + x_{21} + I_0 - d_1$$

$$y_t = x_{1t} + x_{2t} + y_{t-1} - d_t, \quad \forall t = 2,\dots,12$$

$$0 \le x_{1t} \le a_1, \quad \forall t = 1,\dots,12$$

$$0 \le x_{2t} \le a_2, \quad \forall t = 1,\dots,12$$

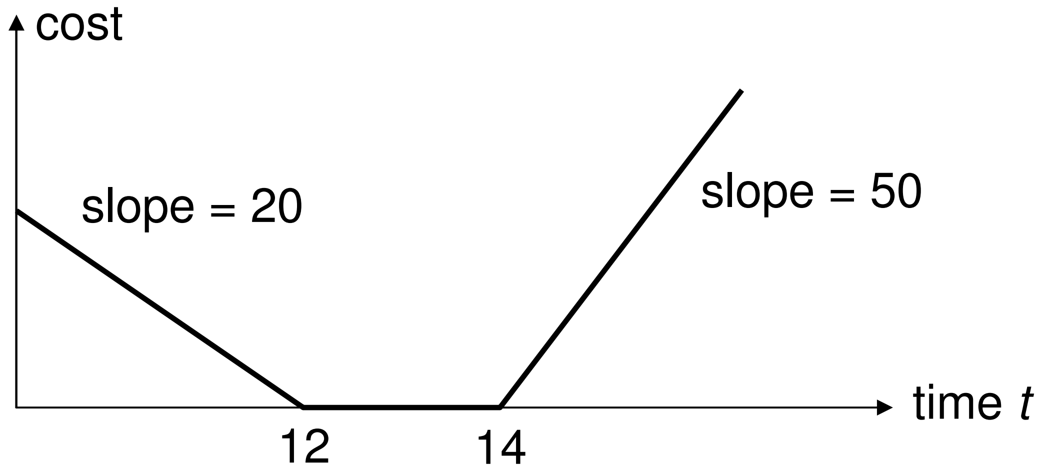
$$0 \le l_t \le L_t, \quad \forall t = 1,\dots,12$$

$$y_t \ge 0, \quad \forall t = 1,\dots,12$$

$$\begin{cases} \min & m \\ s.t. & m \ge |l_t - l_{t-1}|, \quad \forall t = 2,\dots,12 \\ & l_t = b_1 x_{1t} + b_2 x_{2t}, \quad \forall t = 1,\dots,12 \\ & \vdots \\ & etc \end{cases}$$

$$\begin{cases} \min & m \\ s.t. & \left. \begin{array}{l} m \ge l_t - l_{t-1} \\ m \ge l_{t-1} - l_t \end{array} \right\}, \quad \forall t = 2,\dots,12 \\ & l_t = b_1 x_{1t} + b_2 x_{2t}, \quad \forall t = 1,\dots,12 \\ & \vdots \\ & etc \end{cases}$$

Penalties on Soft Constraints



To model as an LP, introduce two new variables

y = # hours early

z = # hours late

via

$$y \geq 12 - t, \; y \geq 0$$

$$z \geq t - 14, \; z \geq 0$$

with objective function

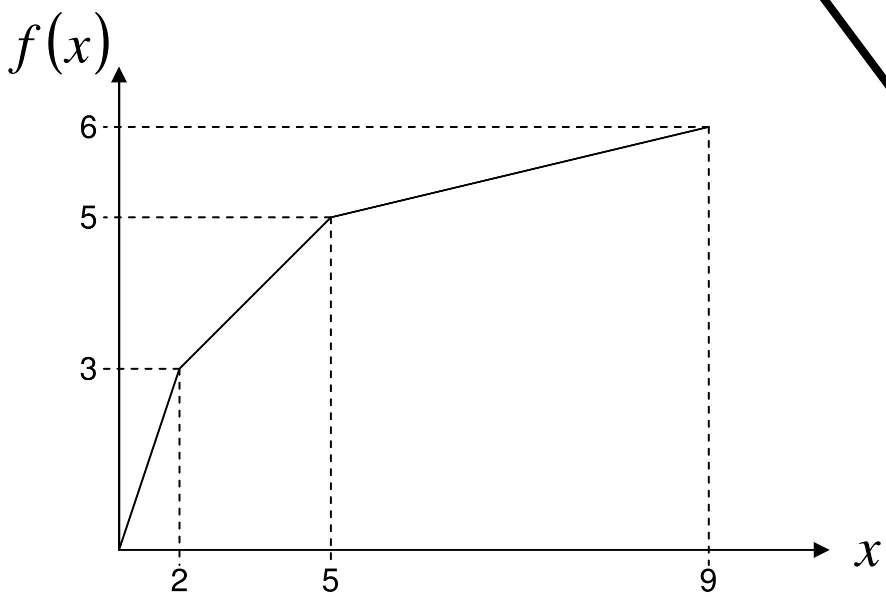
$$\min \ldots + 20y + 50z + \ldots$$

Max. a Piecewise Linear Concave Function

Example:

$$\begin{cases} \max & f(x_1) + 2x_2 \\ s.t. & x_1 + 3x_2 \le 15 \\ & x_1 + x_2 \le 9 \\ & x_1, x_2 \ge 0 \end{cases}$$

where $f(x) = \begin{cases} \frac{3}{2}x, & 0 \le x \le 2 \\ \frac{1}{3}(2x+5), & 2 < x \le 5 \\ \frac{1}{4}(x+15) & 5 < x \le 9 \end{cases}$



$$\begin{cases} \max & \frac{3}{2}y_1 + \frac{2}{3}y_2 + \frac{1}{4}y_3 + 2x_2 \\ s.t. & y_1 + y_2 + y_3 + 3x_2 \le 15 \\ & y_1 + y_2 + y_3 + x_2 \le 9 \\ & 0 \le y_1 \le 2 \\ & 0 \le y_2 \le 3 \\ & 0 \le y_3 \le 4 \\ & x_2 \ge 0 \end{cases}$$

Example: Piecewise Linear Function

To complete the MIP model of the piecewise linear cost function *c*, we note that *c* can be expressed as

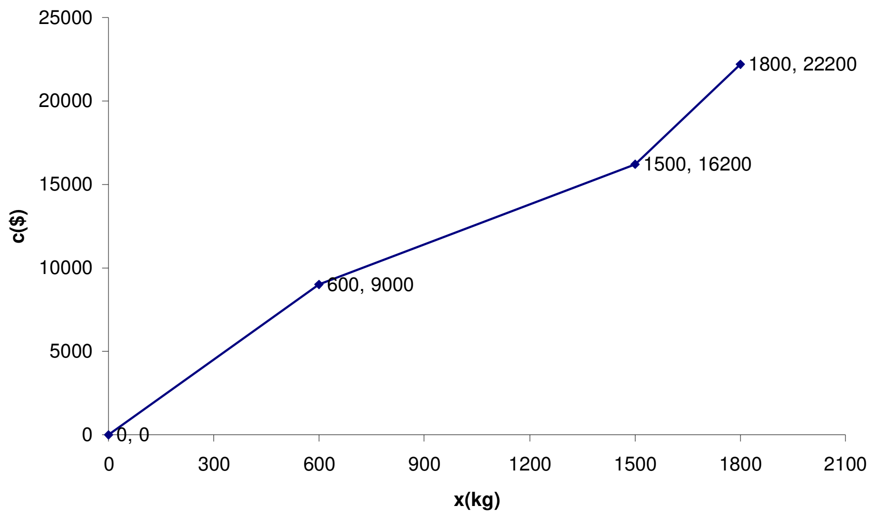
$$c(x)=\sum_{k=1}^{4}\lambda_k c(t_k)$$
$$=0\lambda_1+9000\lambda_2+16200\lambda_3+22200\lambda_4$$
$$=9000\lambda_2+16200\lambda_3+22200\lambda_4$$

$$c(x)=\begin{cases}15x, & 0\le x\le 600\\ 4200+8x & 600\le x\le 1500\\ 20x-13800 & 1500\le x\le 1800\end{cases}$$

s.t.

$$\lambda_1\le y_1$$
$$\lambda_2\le y_1+y_2$$
$$\lambda_3\le y_2+y_3$$
$$\lambda_4\le y_3$$

$$y_1+y_2+y_3=1$$
$$\lambda_1+\lambda_2+\lambda_3+\lambda_4=1$$
$$\lambda_i\ge 0,\quad \forall i=1\ldots 4$$
$$y_i \text{ binary},\quad \forall i=1\ldots 3$$



Linearising the QAP Model

$$x_{ij} = \begin{cases} 1, & \text{flight } i \text{ is assigned to gate } k \\ 0, & otherwise \end{cases}$$

$$y_{ijkl} = \begin{cases} 1, & i \text{ assigned to } k \text{ and } j \text{ to } l \\ 0, & otherwise \end{cases}$$

$$\min \sum_{i \in F} \sum_{j \in F} \sum_{k \in G} \sum_{l \in G} t_{ij} d_{kl} x_{ik} x_{jl}$$

$$s.t.$$

$$\sum_{k \in G} x_{ik} = 1, \quad \forall i \in F$$

$$\sum_{k \in F} x_{ik} = 1, \quad \forall k \in G$$

$$x_{ij} \in \{0,1\}, \quad \forall i \in F, k \in G$$

→

$$\min \sum_{i} \sum_{j} \sum_{k} \sum_{l} t_{ij} d_{kl} y_{ijkl}$$

$$s.t.$$

$$\sum_{i} \sum_{j} \sum_{k} \sum_{l} y_{ijkl} = n^2$$

$$x_{ik} + x_{jl} \geq 2 y_{ijkl}, \quad \forall i, j, k, l$$

$$y_{ijkl} \geq x_{ik} + x_{jl} - 1$$

Project Management

$$\min \quad \sum_{i\in S} C_i x_i + \sum_{k\in H}\sum_{i\in J_k} W_k l_{ik}$$

$$s.t.$$

$$t_i + D_i - x_i \le T, \quad \forall i \in S$$

$$t_i \ge t_j + D_j - x_j, \quad \forall i \in S,\ j \in P_i$$

$$x_i \le D_i, \quad \forall i \in S$$

$$\sum_{\substack{k\in H \\ s.t.\, i\in J_k}} y_{ik} = 1, \quad \forall i \in S$$

$$l_{ik} \ge D_i - x_i - M\left(1 - y_{ik}\right), \quad \forall k \in H, i \in J_k$$

$$t_i, x_i \ge 0, \quad \forall i \in S$$

$$y_{ik} \in \{0,1\}, l_{ik} \ge 0, \quad \forall k \in H, i \in J_k$$

Employee shift covering

$$\min \quad \sum_{q\in S}\sum_{t=1}^{T} C_t x_{tq}$$

$$s.t.$$

$$n_t = \sum_{q\in S}\sum_{r=t-L_q+1}^{t} x_{rq}, \quad \forall t = 1,\ldots,T$$

$$D_t \le n_t \le W, \quad \forall t = 1,\ldots,T$$

$$\sum_{q\in S}\sum_{t=1}^{T} L_q x_{tq} \le H$$

← What is the use of this constraint?

A Formal Definition

- Model M_1 is at least as good as model M_2 if

$$R_1 \subseteq R_2$$

- Model M_1 is better than model M_2 if

$$R_1 \subset R_2$$

- Model M_1 is better than model M_2 with respect to objective **cx** if

$$\max_{x \in R1} \mathbf{cx} < \max_{x \in R2} \mathbf{cx}$$

Uncapacitated Facility Location (UFL)

$$
\begin{aligned}
\min \quad & \sum_{i\in I,\, j\in J} c_{ij} y_{ij} + \sum_{i\in I} f_i x_i \\
s.t. \quad & \sum_{i\in I} y_{ij} = 1, \quad \forall j \in J \\
& \sum_{j\in J} y_{ij} \le m x_i, \quad \forall i \in I \\
& y_{ij} \ge 0, \quad \forall i \in I, j \in J \\
& x_i \in \{0,1\}, \quad \forall i \in I
\end{aligned}
$$

VS

$$
\begin{aligned}
\min \quad & \sum_{i\in I,\, j\in J} c_{ij} y_{ij} + \sum_{i\in I} f_i x_i \\
s.t. \quad & \sum_{i\in I} y_{ij} = 1, \quad \forall j \in J \\
& y_{ij} \le x_i, \quad \forall i \in I, j \in J \\
& y_{ij} \ge 0, \quad \forall i \in I, j \in J \\
& x_i \in \{0,1\}, \quad \forall i \in I
\end{aligned}
$$

Lot-Sizing Problem

$$\min \sum_{t=1}^{T} \left(p_t y_t + h_t s_t + c_t x_t \right)$$

$s.t.$

$$y_1 = d_1 + s_1$$

$$s_{t-1} + y_t = d_t + s_t, \quad t = 2, \ldots, T-1$$

$$s_{T-1} + y_T = d_T$$

$$y_t \le M x_t, \quad t = 1, \ldots, T$$

$$y_t, s_t \ge 0,\ x_t \in \{0,1\}, \quad t = 1, \ldots, T$$

$$M = \sum_{k=1}^{T} d_k$$

A tighter bound:

$$M_t = \sum_{k=t}^{T} d_k$$

VS

$$\min \sum_{t=1}^{T} \left[\left[\sum_{i=1}^{t} \left(p_i + \sum_{k=i}^{t-1} h_k \right) q_{it} \right] + c_t x_t \right]$$

$s.t.$

$$\sum_{i=1}^{T} q_{it} = d_t, \quad t = 1, \ldots, T$$

$$q_{it} \le d_t x_i, \quad i = 1, \ldots, T, t = 1, \ldots, T$$

$$q_{it} \ge 0, \quad i = 1, \ldots, T, t = 1, \ldots, T$$

$$x_i \in \{0,1\}, \quad i = 1, \ldots, T$$

HRP – Assignment Model

$$\min \quad \sum_{j=1}^{m} w_j$$

$$s.t.$$

$$\sum_{j=1}^{m} z_{ij} = 1, \quad \forall i = 1,\ldots,n$$

$$\sum_{i=1}^{n} h_i z_{ij} \le H w_j, \quad \forall j = 1,\ldots,m$$

$$z_{ij} \in \{0,1\}, \quad \forall i = 1,\ldots,n;\ j = 1,\ldots,m$$

$$w_j \in \{0,1\}, \quad \forall j = 1,\ldots,m$$

VS

$$\min \quad \sum_{s \in J} x_s$$

$$s.t.$$

$$\sum_{s \in J : i \in S} x_s = 1, \quad \forall i = 1,\ldots,n$$

$$x_s \in \{0,1\}, \quad \forall s \in J$$

…might have other restrictions, e.g. some jobs cannot be done by the same worker.

Any problems?
SYMMETRY – every worker is identical

(May also have symmetry if several jobs require same length of time.)

HRP – Assignment Model

How can we break symmetry?

One way is to include the following set of constraints:

$$\sum_{i=1}^{n} z_{i(j-1)} \geq \sum_{i=1}^{n} z_{ij}, \quad \forall j = 2, \ldots, m$$

Considers only solutions where the number of jobs assigned to a worker is non-decreasing with increasing worker "index".

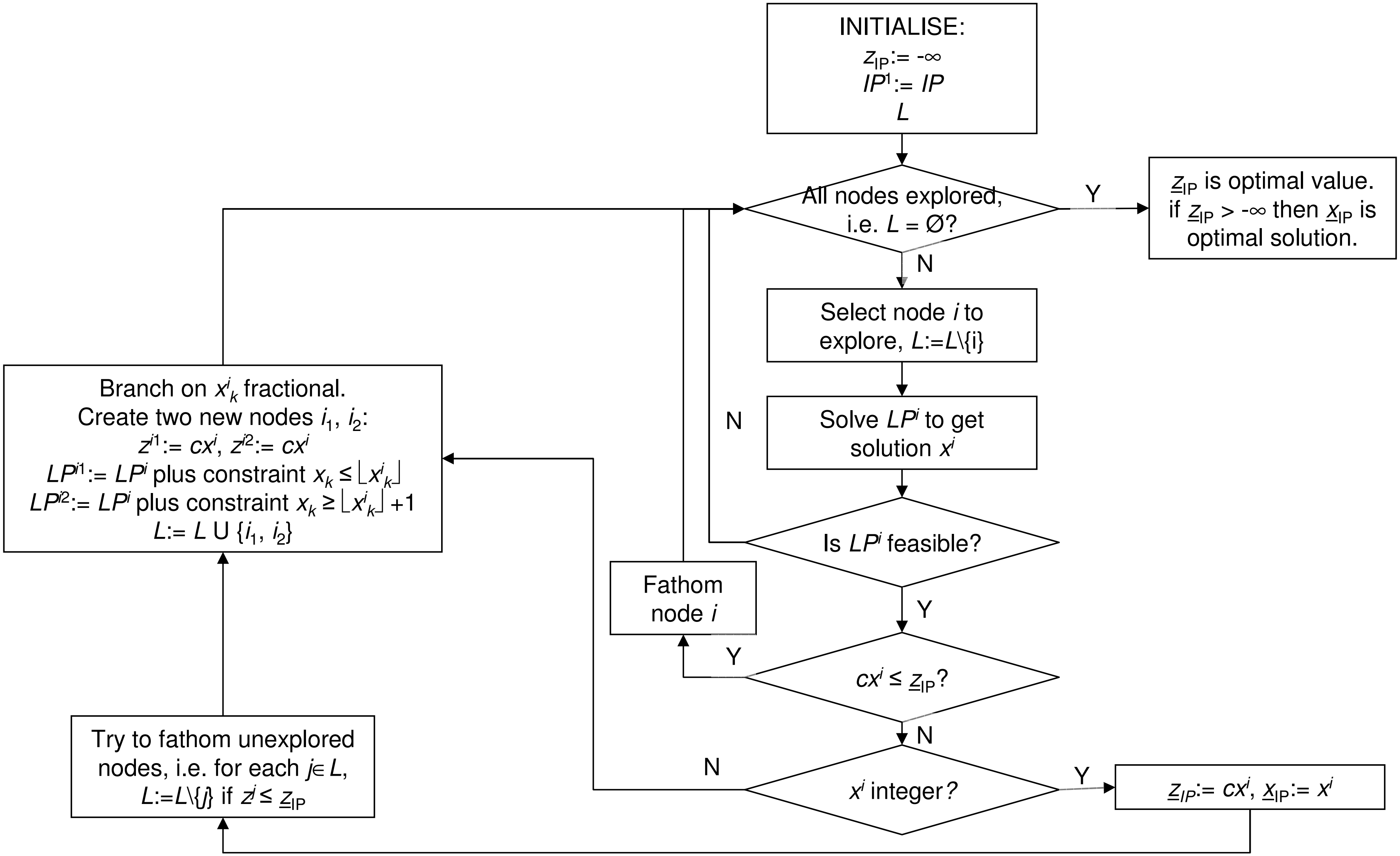
- This is reasonable since any feasible solution obtained for this problem can be arranged in this order.

Also…

review other MIP modelling problems (lectures and practice classes)

The Branch-and-Bound Method

Branch-and-Bound Algorithm (Max. prob.)



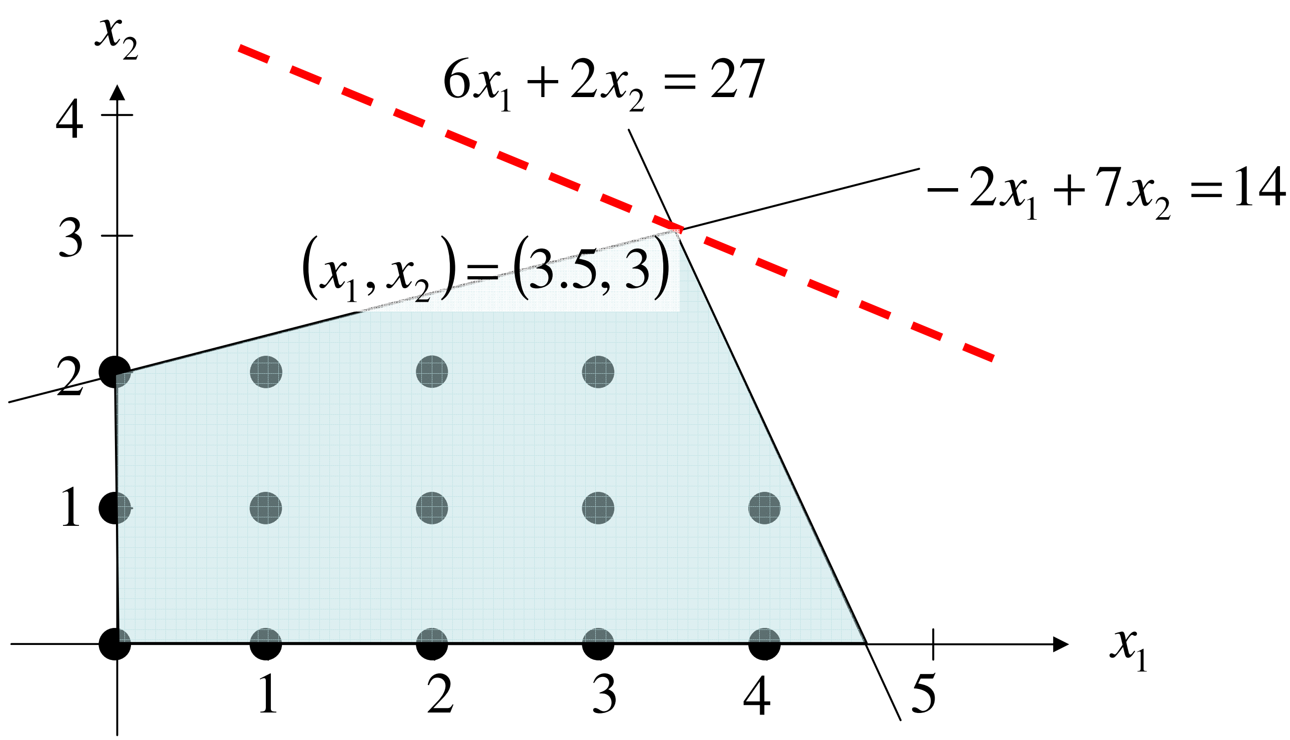
LP-based Branch-and-Bound

Example 1

$$\max\ z = x_1 + 2x_2$$
$$\text{s.t.} -2x_1 + 7x_2 \le 14$$
$$6x_1 + 2x_2 \le 27$$
$$x_1, x_2 \text{ integer}$$

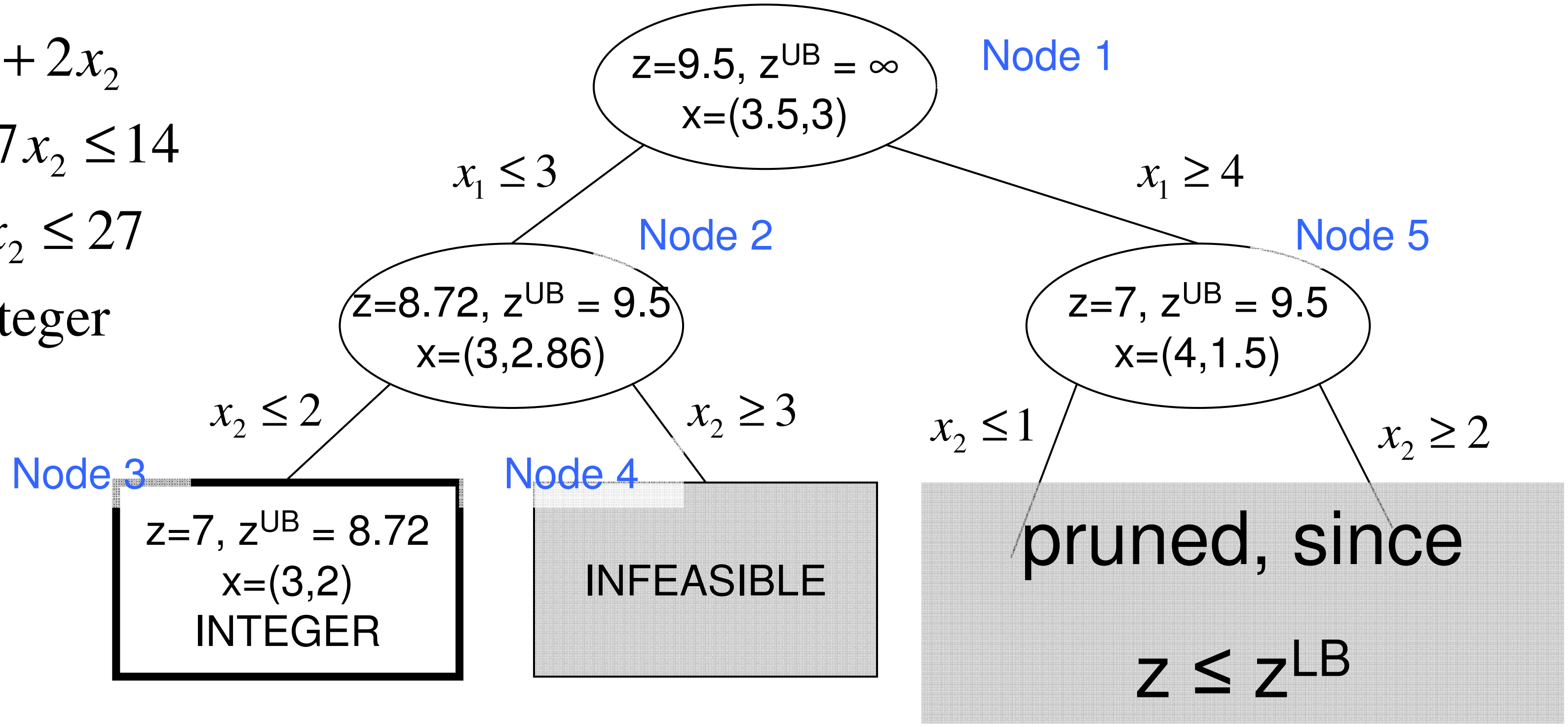
Best objective = *NONE*

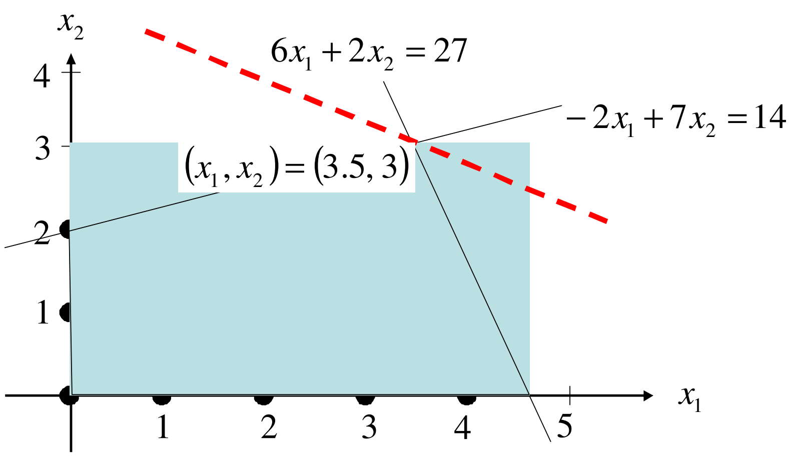
$\bar{z} = 9.5$



Example 1: B&B Tree

$$\max\ z = x_1 + 2x_2$$
$$\text{s.t.}\ -2x_1 + 7x_2 \le 14$$
$$6x_1 + 2x_2 \le 27$$
$$x_1, x_2 \text{ integer}$$





z^{LB}

initial: $-\infty$

update: 7

The Gomory Cutting Plane Method

The Gomory Cutting Plane Method

Proposition 1.

If $\alpha x \leq \beta$ is valid for P then

$$\sum_{i=1}^{n} \lfloor \alpha_i \rfloor x_i \leq \lfloor \beta \rfloor \text{ is also valid for } P.$$

The Chavátal-Gomory Procedure

…is determined by the following proposition

Proposition 2.

Let $P = \{x \in Z^n : Ax \le b, x \ge 0\}$, where $A = (a_{ij}) \in \Re^{m\mathrm{x}n}$, and $u \in \Re^m_+$.

Then

$$\sum_{j=1}^{n} \left\lfloor \sum_{i=1}^{n} u_i a_{ij} \right\rfloor x_j \le \left\lfloor \sum_{i=1}^{m} u_i b_i \right\rfloor$$

or equivalently

$$\sum_{j=1}^{n} \lfloor u^T a_j \rfloor x_j \le \lfloor u^T b \rfloor$$

where a_j is j^{th} column of A, is valid constraint for P.

Proposition 3

If $\alpha x = \beta$ is valid for P then

$$\sum_{i=1}^{n} (\alpha_i - \lfloor \alpha_i \rfloor) x_i \geq \beta - \lfloor \beta \rfloor$$

is also valid for P.

Proof. $\alpha x \leq \beta$ is obviously valid, so by Proposition 1,

$$\sum_{i=1}^{n} \lfloor \alpha_i \rfloor x_i \leq \lfloor \beta \rfloor, \quad \forall x \in P$$

$$\Leftrightarrow \quad \sum_{i=1}^{n} \alpha_i x_i - \sum_{i=1}^{n} \lfloor \alpha_i \rfloor x_i \geq \beta - \lfloor \beta \rfloor, \quad \forall x \in P$$

$$\Leftrightarrow \quad \sum_{i=1}^{n} (\alpha_i - \lfloor \alpha_i \rfloor) x_i \geq \beta - \lfloor \beta \rfloor, \quad \forall x \in P$$

The Dual Simplex Method

Initial tableau:

Eqn	x_N	x_B	RHS
1			
$\vdots$	y	I	r
m			
z	y_0	0	r_0

Assumption: $y_0 \geq 0$ (if max. problem); $y_0 \leq 0$ (if min. problem)

1. If $r \geq 0$ the STOP: basic feasible solution is <u>optimal</u>.
   Else select row i such that $r_i = \min_j\{r_j\}$ (i.e. select row with the most negative RHS).

2. If $y_{ij} \geq 0$ for all j then STOP: dual is unbounded, so primal is <u>infeasible</u>.
   Else choose $j \in N$ such that $j = \arg\min\{|y_{0j}/y_{ij}| : j \in N, y_{ij} < 0\}$

3. Pivot so that j enters the basis and i leaves the basis. Go to Step 1.

Example 3 (repeat example)

$$\max \quad z = 7x_1 + 9x_2$$
$$s.t. \quad -x_1 + 3x_2 \le 6$$
$$7x_1 + x_2 \le 35$$
$$x_1, x_2 \in Z_+$$

Final tableau after Simplex method:

eqn	x1	x2	s1	s2	RHS	Ratio test	
1	0	1	7/22	1/22	3 1/2		R1=R1+R2'/3
2	1	0	- 1/22	3/22	4 1/2		R2'=R2*3/22
z	0	0	2 6/11	1 4/11	63		Rz=Rz+10*R2'

Gomory cut due to row 1 is $7/22s_1 + 1/22s_2 \ge 1/2$

Example 3 (repeat example)

Add cut $7/22s_1 + 1/22s_2 \geq 1/2$ to tableau.

Applying dual simplex to the new tableau:

eqn	x1	x2	s1	s2	s3	RHS
1	0	1	7/22	1/22	0	3 1/2
2	1	0	- 1/22	3/22	0	4 1/2
3	0	0	- 7/22	- 1/22	1	- 1/2
z	0	0	2 6/11	1 4/11	0	63
Ratio Test			8	30		

eqn	x1	x2	s1	s2	s3	RHS	
1	0	1	0	0	1	3	R1=R1-R3'*7/22
2	1	0	0	1/7	- 1/7	4 4/7	R2=R2-R3'/22
3	0	0	1	1/7	-3 1/7	1 4/7	R3'=R3*-22/7
z	0	0	0	1	8	59	Rz=Rz-R3'28/11
Ratio Test							

Optimal solution: $x_1 = 32/7$, $x_2 = 3$, $z = 59$

Gomory cut due to row 2 is $1/7s_2 + 6/7s_3 \geq 4/7$

Example 3 (repeat example)

Add cut $1/7s_2 + 6/7s_3 \geq 4/7$ to tableau.

Applying dual simplex to the new tableau:

eqn	x1	x2	s1	s2	s3	s4	RHS
1	0	1	0	0	1	0	3
2	1	0	0	1/7	- 1/7	0	4 4/7
3	0	0	1	1/7	-3 1/7	0	1 4/7
4	0	0	0	- 1/7	- 6/7	1	- 4/7
z	0	0	0	1	8	0	59
Ratio Test				7	9.3333333		

eqn	x1	x2	s1	s2	s3	s4	RHS	
1	0	1	0	0	1	0	3	R1=R1
2	1	0	0	0	-1	1	4	R2=R2-1/7*R4'
3	0	0	1	0	-4	1	1	R3=R3-1/7*R4'
4	0	0	0	1	6	-7	4	R4'=R4*-7
z	0	0	0	0	2	7	55	Rz=Rz-R4'

Optimal solution: $x_1 = 4$, $x_2 = 3$, $z = 55$ (integer!!)

Column Generation and Dantzig-Wolfe Decomposition

Algorithm to solve the Master LP

Step 0:

Choose an initial set of columns A^s.

Step 1:

Solve the Restricted LP to obtain optimal basic solution *y* and corresponding duals *u*.

Step 2:

Note that *x* given by

$$x_i = y_i, \text{ if column } i \text{ is in } A^s; 0, \text{ otherwise}$$

is a basic feasible point for the Master LP and is optimal if

$$uA \leq c.$$

Solve column generation subproblem:

$$z^{sub} = \min\{c_i - ua_i : a_i \text{ is a column of } A\}$$

If $z^{sub} \geq 0$ then STOP: *x* is optimal for Master LP.

Otherwise add a_i such that $c_i - ua_i = z^{sub}$ to A^s and go to Step 1.

Human Resource Planning (HRP)

Tasks to be performed: 1, 2, …, n

h_i = time required for task i (hours)

H = max number of hours worker can work in a day

What is the minimum workforce required?

Master IP:

$$z = \min \sum_{s \in J'} x_s$$

$$s.t.$$

$$\sum_{s \in J' : i \in s} x_s = 1, \quad \forall i = 1, \ldots, n$$

$$x_s \in \{0,1\}, \quad \forall s \in J'$$

Subproblem:

$$z^{sub} = 1 - \begin{cases} \max \sum_{i=1}^{n} u_i y_i \\ s.t. \\ \sum_{i=1}^{n} h_i y_i \leq H \\ y_i \in \{0,1\}, \quad \forall i = 1, \ldots, n \end{cases}$$

DW Decomposition…

$$z = \max \quad \mathbf{C}^1\mathbf{x}^1 + \mathbf{C}^2\mathbf{x}^2$$

$$s.t.$$

$$\mathbf{D}^1\mathbf{x}^1 \le \mathbf{d}^1$$

$$\mathbf{D}^2\mathbf{x}^2 \le \mathbf{d}^2$$

$$\mathbf{A}^1\mathbf{x}^1 + \mathbf{A}^2\mathbf{x}^2 \le \mathbf{b}$$

$$\equiv$$

$$z = \max \quad \mathbf{C}^1\left(\sum_{t=1}^{|\mathbf{T}_1|} \lambda_t^1 \mathbf{x}_t^1\right) + \mathbf{C}^2\left(\sum_{t=1}^{|\mathbf{T}_2|} \lambda_t^2 \mathbf{x}_t^2\right)$$

$$s.t.$$

$$\sum_{t=1}^{|\mathbf{T}_1|} \lambda_t^1 = 1$$

$$\sum_{t=1}^{|\mathbf{T}_2|} \lambda_t^2 = 1$$

$$\mathbf{A}^1\left(\sum_{t=1}^{|\mathbf{T}_1|} \lambda_t^1 \mathbf{x}_t^1\right) + \mathbf{A}^2\left(\sum_{t=1}^{|\mathbf{T}_2|} \lambda_t^2 \mathbf{x}_t^2\right) \le \mathbf{b}$$

$$\lambda_t^1 \ge 0, \quad t = 1, \ldots, |\mathbf{T}_1|$$

$$\lambda_t^2 \ge 0, \quad t = 1, \ldots, |\mathbf{T}_2|$$

Pricing subproblems…

Restricted Master LP

$$z = \max \quad \mathbf{C}^1\left(\sum_{t=1}^{|\hat{\mathbf{T}}_1|} \lambda_t^1 \mathbf{x}_t^1\right) + \mathbf{C}^2\left(\sum_{t=1}^{|\hat{\mathbf{T}}_2|} \lambda_t^2 \mathbf{x}_t^2\right)$$

$s.t.$

$$\sum_{t=1}^{|\hat{\mathbf{T}}_1|} \lambda_t^1 = 1 \quad (dual: \pi_1)$$

$$\sum_{t=1}^{|\hat{\mathbf{T}}_2|} \lambda_t^2 = 1 \quad (dual: \pi_2)$$

$$\mathbf{A}^1\left(\sum_{t=1}^{|\hat{\mathbf{T}}_1|} \lambda_t^1 \mathbf{x}_t^1\right) + \mathbf{A}^2\left(\sum_{t=1}^{|\hat{\mathbf{T}}_2|} \lambda_t^2 \mathbf{x}_t^2\right) \leq \mathbf{b} \quad (dual: \rho)$$

$$\lambda_t^1 \geq 0, \quad t = 1, \ldots, |\hat{\mathbf{T}}_1|$$

$$\lambda_t^2 \geq 0, \quad t = 1, \ldots, |\hat{\mathbf{T}}_2|$$

Pricing subproblem for x^1

$$\max \quad z_1 = \mathbf{C}^1\mathbf{x}^1 - \rho\mathbf{A}^1\mathbf{x}^1 - \pi_1$$

$s.t.$

$$\mathbf{D}^1\mathbf{x}^1 \leq \mathbf{d}^1$$

$$\mathbf{x}^1 \geq 0$$

Pricing subproblem for x^2

$$\max \quad z_2 = \mathbf{C}^2\mathbf{x}^2 - \rho\mathbf{A}^2\mathbf{x}^2 - \pi_2$$

$s.t.$

$$\mathbf{D}^2\mathbf{x}^2 \leq \mathbf{d}^2$$

$$\mathbf{x}^2 \geq 0$$

Quadratic Models

Portfolio Optimisation: Xpress^{MP} Example

An investor is evaluating ten different securities (`shares').

He estimates the return on investment for a period of one year.

He further wishes to invest at least half of his capital in North-American shares and at most a third in shares.

How should the capital be divided among the shares to minimize the risk whilst obtaining a certain target yield?

The investor adopts the **Markowitz** idea of getting estimates of the variance/covariance matrix of estimated returns on the securities.

Portfolio Optimisation: Xpress^{MP} Example

Given

I^{max} = max. investment per share (%)

I^{min} = min. investment into North American share (%)

S = set of all shares

A = set of North American shares

T = target yield (%)

R_p = estimated return in investment for share *p* (%)

Cov = covariance matrix of estimated return

Variable:

x_p = fraction of capital invested in share *p*

Small Example

Budget = \$2,000 (spend all)

Minimum return required = 5%

$$
\begin{cases}
\min \quad \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{bmatrix} 26.25 & 2.1 \\ 2.1 & 36.04 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
s.t. \\
4.5x_1 + 5.6x_2 \geq 10000 \\
x_1 + x_2 = 2000 \\
x_1, x_2 \geq 0
\end{cases}
$$

$$\frac{(4.5x_1 + 5.6x_2)/100}{2000} \geq \frac{5}{100}$$

Portfolio Optimisation: Xpress^{MP} Example

$$\min \quad \sum_{p \in S} \sum_{q \in S} \operatorname{cov}(p,q) x_p x_q$$

$$s.t.$$

$$\sum_{p \in A} x_p \geq I^{\min}$$

$$\sum_{p \in S} x_p = 1$$

$$x_p \leq I^{\max}, \quad \forall p \in S$$

$$\sum_{p \in S} R_p x_p \geq T$$

$$x_p \geq 0, \quad \forall p \in S$$