

620-362
Applied Operations Research

An Introduction
to Integer Programming Modelling

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Some contents of this presentation are adapted from year 2005 course notes for 620-362 Applied Operations Research, Department of Mathematics and Statistics, The University of Melbourne (compiled by Prof Natasha Boland and Dr Renata Sotirov)

Integer Programming Modelling

- Knapsack Problems (Capital Budgeting)
- Set Covering/Partitioning/Packing (Human resource planning)
- Facility Location Problems
- Lot-sizing Problems
- Logic and Disjunctive Constraints
- Piecewise Linear Functions
- Quadratic Assignment Problems
- Hub Location Problems (Postal Service Planning)
- Vehicle Routing/Travelling Salesperson

Capital Budgeting

Stockco is considering 4 investments.

Investment	Outlay Now	Yield (NPV)
1	\$5,000	\$16,000
2	\$7,000	\$22,000
3	\$4,000	\$12,000
4	\$3,000	\$8,000

The yield calculation presumes that the entire outlay is invested; it is not possible to invest, for example, only \$2,500 in Investment 1 and expect a commensurate return.

Stockco has an investment budget of \$14,000.

Which investments should Stockco make in order to maximise their total yield?

Note: $16/5 > 22/7 > 12/4 > 8/3$

So maybe we should take Investments 1 and 2 and get a yield of \$38,000?

Capital Budgeting

Let

$$x_j = \begin{cases} 1, & \text{if investment } j \text{ is made} \\ 0, & \text{otherwise} \end{cases}, \text{ for } j = 1, \dots, 4$$

Stockco's problem can be modelled as

$$\text{maximise} \quad 16x_1 + 22x_2 + 12x_3 + 8x_4$$

$$\text{s.t.} \quad 5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14$$

$$x_j \text{ binary, } j = 1, \dots, 4$$

This is known as a **Binary Knapsack Problem**.

Capital Budgeting

“Greedy approach”:

take Investments 1 and 2, value = \$38,000

LP-solution:

take Investments 1 and 2 and half of Investment 3,
value = \$44,000

Optimal integer solution:

take Investments 2, 3 and 4, value = \$42,000

0-1 Knapsack Problem

S = set of projects

p_j = profit of carrying out project i

w_j = cost of carrying out project i

b = budget available

Objective:

Maximum overall profit, such that the total cost of carrying out the projects is within the budget.

0-1 Knapsack Problem

$$x_j = \begin{cases} 1, & \text{project } j \text{ is selected} \\ 0, & \text{otherwise} \end{cases}$$

$$\textit{Maximize} \quad \sum_{j \in S} p_j x_j$$

$$\textit{s.t.} \quad \sum_{j \in S} w_j x_j \leq b$$

$$x_j \in \{0,1\}, \forall j \in S$$

Modelling Logic And Disjunctive Constraints

Logical statements, such as implications, can be modelled using binary variables.

Facility location problem: no more than 10 new facilities to be constructed

$$\sum_{i=1}^m y_i \leq 10$$

Can put facilities at both sites 1 and 2 or at neither

$$y_1 = y_2$$

Cannot put facilities at both sites 3 and 4

$$y_3 + y_4 \leq 1$$

If put facility at site 5, must also put facility at site 6

$$y_5 \leq y_6$$

Disjunctive Inequalities

Machine maintenance scheduling:

- Schedule time t to perform maintenance on machine
- Machine will be in use between time T_1 and T_2
- Deadline on maintenance is time T

Disjunctive constraint:

$$t \leq T_1 \quad \text{or} \quad t \geq T_2$$

$$y = \begin{cases} 1, & t \leq T_1 \text{ is enforced} \\ 0, & t \geq T_2 \text{ is enforced} \end{cases}$$

$$t \leq T_1 + (T - T_1)(1 - y)$$

$$t \geq T_2 - T_2 y$$

Modelling Logical Relations

Select at most one (packing): $\sum_{j \in S} x_j \leq 1$

Select precisely one (partitioning): $\sum_{j \in S} x_j = 1$

Select at least one (covering): $\sum_{j \in S} x_j \geq 1$

Special conditions:

$$x_1 = 1 \Rightarrow x_2 = 1$$

$$x_1 - x_2 \leq 0$$

$$x_1 = 1 \Rightarrow x_2 = 0$$

$$x_1 + x_2 \leq 1$$

$$x_1 = 1 \wedge x_2 = 1 \Rightarrow x_3 = 1$$

$$x_1 + x_2 - x_3 \leq 1$$

Stocko examples...

Stocko can make at most two investments

$$x_1 + x_2 + x_3 + x_4 \leq 2$$

If Stocko invest in investment #2 then they must also invest in #1; if $x_2 = 1$ then $x_1 = 1$

$$x_1 \geq x_2$$

If Stocko invest in #2, they can't invest in #3; if

$$x_2 + x_3 \leq 1$$

Min-value

Variables: x, u_1, u_2 all continuous

$$x = \min(u_1, u_2)$$

$$x \leq u_1, x \leq u_2 \text{ and } \underline{(x \geq u_1 \text{ or } x \geq u_2)}$$

Feasible set bounded:

$$m1 \geq u_1 - x$$

for all feasible x, u_1

$$m2 \geq u_2 - x$$

for all feasible x, u_2

Min-value

$$y = \begin{cases} 1, & x \geq u_1 \text{ is enforced} \\ 0, & x \geq u_2 \text{ is enforced} \end{cases}$$

$$x \geq u_1 - m_1(1 - y)$$

$$x \geq u_2 - m_2 y$$

y binary

Example: Piecewise Linear Function

Shipping a commodity:

small quantities -> high cost per unit shipped

large quantities -> lower cost per unit shipped

- Maximum shipment 1800kg
- First 600kg costs \$15 per kg
- Next 900kg costs \$8 per kg
- Normal shipment limit: 1500kg
- Penalties for excess: weight over 1500kg \$20 per kg

Model cost of shipment as piecewise linear function of weight to be shipped.

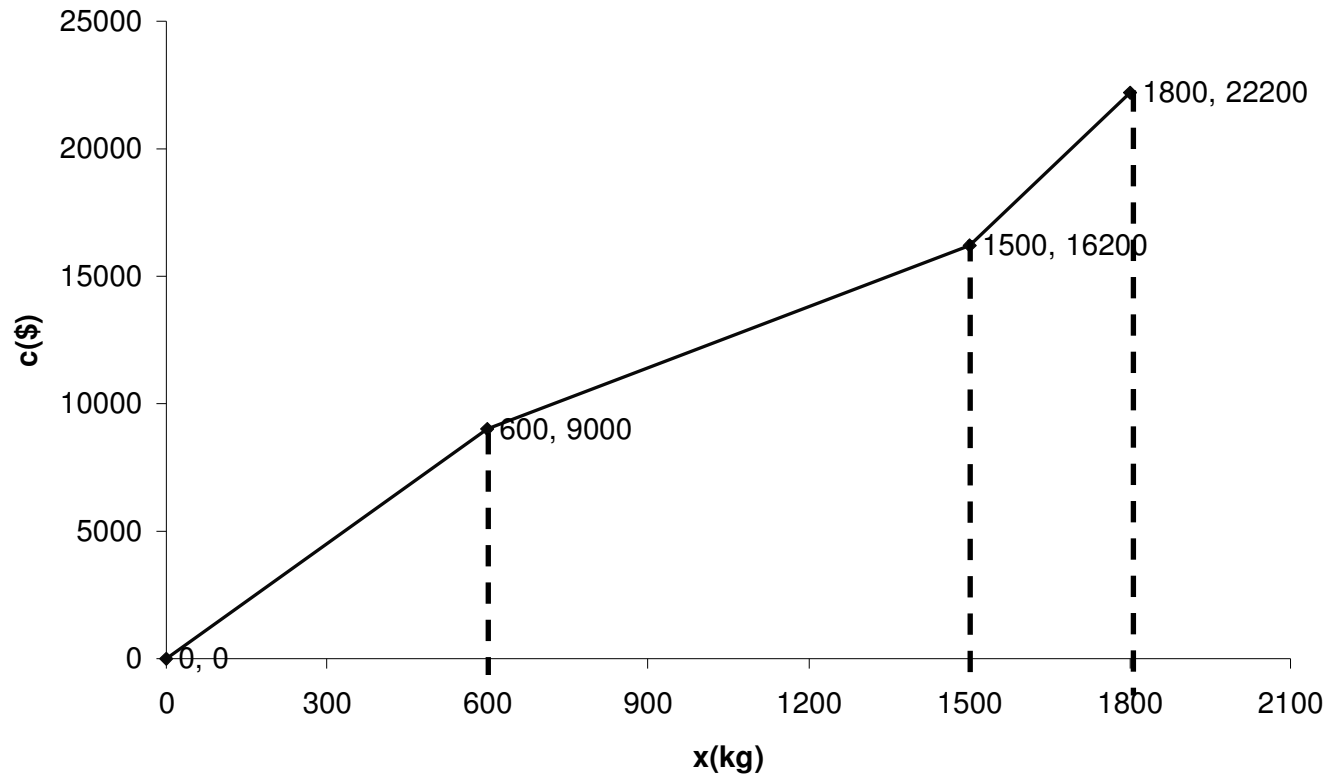
Example: Piecewise Linear Function

x = weight in kg shipped

Cost of shipment in dollars:

$$c(x) = \begin{cases} 15x, & 0 \leq x \leq 600 \\ 9000 + 8(x - 600) & 600 \leq x \leq 1500 \\ 16200 + 20(x - 1500) & 1500 \leq x \leq 1800 \end{cases}$$
$$= \begin{cases} 15x, & 0 \leq x \leq 600 \\ 4200 + 8x & 600 \leq x \leq 1500 \\ 20x - 13800 & 1500 \leq x \leq 1800 \end{cases}$$

Example: Piecewise Linear Function



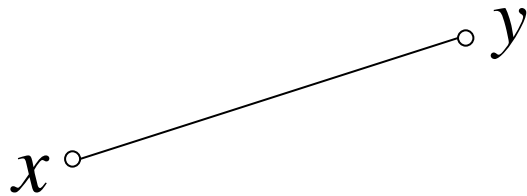
Breakpoints of function:

$$t_1 = 0, t_2 = 600, t_3 = 1500, t_4 = 1800$$

Function is *not differentiable*, *not convex*, *not concave*
– difficult for optimization

Some background

1. Describing a line segment between two points



$$x, y \in \mathfrak{R}^n$$

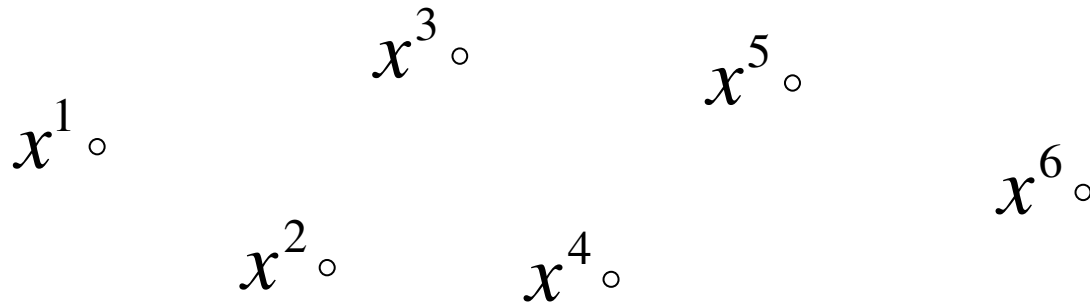
Points on line segment are points of the form

$$\lambda x + (1-\lambda)y, \quad 0 \leq \lambda \leq 1$$

$$\text{or } \lambda_1 x + \lambda_2 y, \quad \lambda_1, \lambda_2 \geq 0, \lambda_1 + \lambda_2 = 1$$

Some background

2. Describing convex hull of points



$$x^1, x^2, \dots, x^k \in \mathcal{R}^n$$

Points in **convex hull** are points of the form

$$\lambda_1 x^1 + \lambda_2 x^2 + \dots + \lambda_k x^k,$$

$$\lambda_1, \lambda_2, \dots, \lambda_k \geq 0,$$

$$\lambda_1 + \lambda_2 + \dots + \lambda_k = 1$$

Such points are called convex combination of x^1, x^2, \dots, x^k .

Example: Piecewise Linear Function

$$x \in [t_k, t_{k+1}] \Rightarrow x = \lambda_k t_k + \lambda_{k+1} t_{k+1}$$

for some $\lambda_k, \lambda_{k+1} \geq 0$ satisfying

$$\lambda_k + \lambda_{k+1} = 1$$

If only two adjacent λ_k nonzero and sum to 1

$$x = \sum_{k=1}^4 \lambda_k t_k$$

Sum to 1:

$$\sum_{k=1}^4 \lambda_k = 1$$

Example: Piecewise Linear Function

Only two adjacent λ_k nonzero:

$$y_i = \begin{cases} 1, & t_i \leq x \leq t_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

$$\lambda_1 \leq y_1$$

$$\lambda_2 \leq y_1 + y_2$$

$$\lambda_3 \leq y_2 + y_3$$

$$\lambda_4 \leq y_3$$

$$y_1 + y_2 + y_3 = 1$$

Example: Piecewise Linear Function

To complete the MIP model of the piecewise linear cost function c , we note that c can be expressed as

$$\begin{aligned}c(x) &= \sum_{k=1}^4 \lambda_k c(t_k) \\ &= 0\lambda_1 + 9000\lambda_2 + 16200\lambda_3 + 22200\lambda_4 \\ &= 9000\lambda_2 + 16200\lambda_3 + 22200\lambda_4\end{aligned}$$

s.t.

$$\begin{array}{ll}\lambda_1 \leq y_1 & y_1 + y_2 + y_3 = 1 \\ \lambda_2 \leq y_1 + y_2 & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1 \\ \lambda_3 \leq y_2 + y_3 & \lambda_i \geq 0, \quad \forall i = 1 \dots 4 \\ \lambda_4 \leq y_3 & y_i \text{ binary}, \quad \forall i = 1 \dots 3\end{array}$$