620-362
Applied Operations Research

Dantzig-Wolfe Decomposition

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Some contents of this presentation are adapted from year 2005 course notes for 620-362 Applied Operations Research, Department of Mathematics and Statistics, The University of Melbourne (compiled by Prof Natashia Boland and Dr Renata Sotirov)

Example

$$
\begin{array}{lllllllllll}
\max & z = & 90x_1 & + & 80x_2 & + & 70x_3 & + & 60x_4 & & \\
s.t. & & 3x_1 & + & x_2 & & & & & \leq & 12 \\
 & & 2x_1 & + & x_2 & & & & & \leq & 10 \\
 & & & & & & 3x_3 & + & 2x_4 & \leq & 15 \\
 & & & & & & x_3 & + & x_4 & \leq & 4 \\
 & & 8x_1 & + & 6x_2 & + & 7x_3 & + & 5x_4 & \leq & 80 \\
 & & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0
\end{array}
$$

Partitioning…

$$
\begin{array}{lcrcrcrcrcr}
\max & z = & 90x_1 & + & 80x_2 & + & 70x_3 & + & 60x_4 & & \\
\hline
s.t. & & 3x_1 & + & x_2 & & & & & \leq & 12 \\
 & & 2x_1 & + & x_2 & & & & & \leq & 10 \\
\hline
 & & & & & & 3x_3 & + & 2x_4 & \leq & 15 \\
 & & & & & & x_3 & + & x_4 & \leq & 4 \\
\hline
 & & 8x_1 & + & 6x_2 & + & 7x_3 & + & 5x_4 & \leq & 80 \\
 & & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0
\end{array}
$$

Partitioning…

$$\mathbf{C}^1\mathbf{x}^1 + \mathbf{C}^2\mathbf{x}^2$$

$$\begin{array}{llcccccccccl}
\max & z = & 90x_1 & + & 80x_2 & + & 70x_3 & + & 60x_4 & & & \\
\hline
s.t. & \mathbf{D}^1\mathbf{x}^1 & 3x_1 & + & x_2 & & & & & \leq & 12 & \mathbf{d}^1 \\
 & & 2x_1 & + & x_2 & & & & & \leq & 10 & \\
\hline
 & \mathbf{D}^2\mathbf{x}^2 & & & & & 3x_3 & + & 2x_4 & \leq & 15 & \mathbf{d}^2 \\
 & & & & & & x_3 & + & x_4 & \leq & 4 & \\
\hline
\multicolumn{2}{l}{\mathbf{A}^1\mathbf{x}^1 + \mathbf{A}^2\mathbf{x}^2} & 8x_1 & + & 6x_2 & + & 7x_3 & + & 5x_4 & \leq & 80 & \mathbf{b} \\
 & & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0 &
\end{array}$$

Partitioning…

$$z = \max \quad \mathbf{C}^1\mathbf{x}^1 + \mathbf{C}^2\mathbf{x}^2$$

$$s.t.$$

$$\mathbf{D}^1\mathbf{x}^1 \le \mathbf{d}^1$$

$$\mathbf{D}^2\mathbf{x}^2 \le \mathbf{d}^2$$

$$\mathbf{A}^1\mathbf{x}^1 + \mathbf{A}^2\mathbf{x}^2 \le \mathbf{b}$$

$$\mathbf{x}^1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \mathbf{x}^2 = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$$

$$\mathbf{D}^1 = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \qquad \mathbf{D}^2 = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

$$\mathbf{A}^1 = \begin{bmatrix} 8 & 6 \end{bmatrix} \qquad \mathbf{A}^2 = \begin{bmatrix} 7 & 5 \end{bmatrix}$$

$$\mathbf{C}^1 = \begin{bmatrix} 90 & 80 \end{bmatrix} \qquad \mathbf{C}^2 = \begin{bmatrix} 70 & 60 \end{bmatrix}$$

$$\mathbf{d}^1 = \begin{bmatrix} 12 \\ 10 \end{bmatrix} \quad \mathbf{d}^2 = \begin{bmatrix} 15 \\ 4 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 80 \end{bmatrix}$$

Minkowski's Theorem

…for bounded feasible region

Let $\mathbf{T}_k = \{\mathbf{x}_1^k, \mathbf{x}_2^k, \ldots\}$ be all extreme points of $\mathbf{D}^k \mathbf{x}^k \leq \mathbf{d}^k$

Then

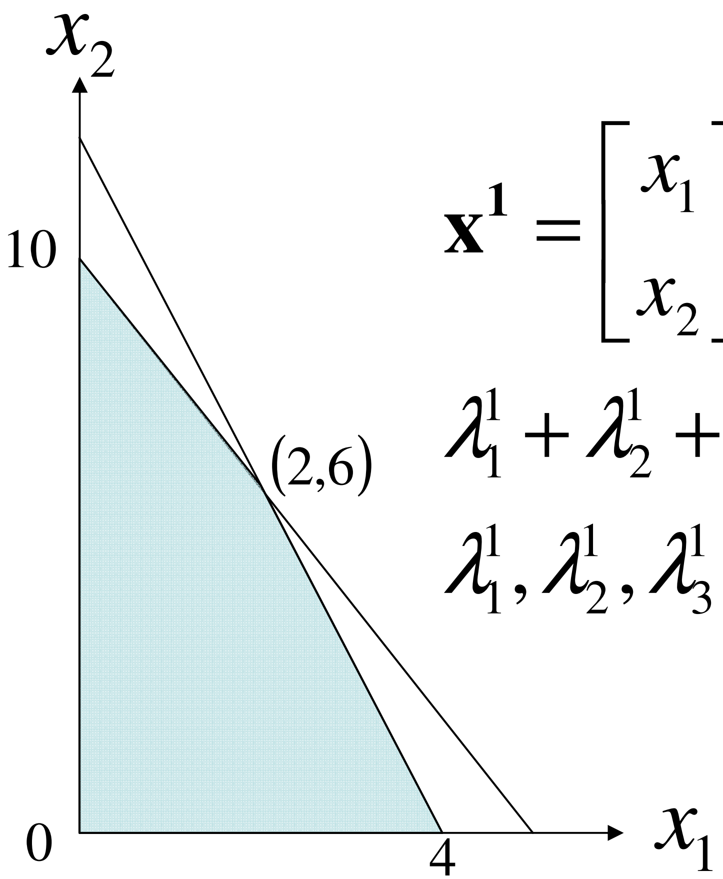
$$\mathbf{x}^k = \lambda_1^k \mathbf{x}_1^k + \lambda_2^k \mathbf{x}_2^k + \ldots + \lambda_{|\mathbf{T}_k|}^k \mathbf{x}_{|\mathbf{T}_k|}^k = \sum_{t=1}^{|\mathbf{T}_k|} \lambda_t^k \mathbf{x}_t^k$$

$$\lambda_1^k + \lambda_2^k + \ldots + \lambda_{|\mathbf{T}_k|}^k = 1$$

$$\lambda_1^k, \lambda_2^k, \ldots, \lambda_{|\mathbf{T}_k|}^k \geq 0$$

Extreme points…

$$\mathbf{D}^1 = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \quad \mathbf{x}^1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \mathbf{d}^1 = \begin{bmatrix} 12 \\ 10 \end{bmatrix} \quad \Rightarrow \begin{matrix} 3x_1 + x_2 \le 12 \\ 2x_1 + x_2 \le 10 \end{matrix}$$



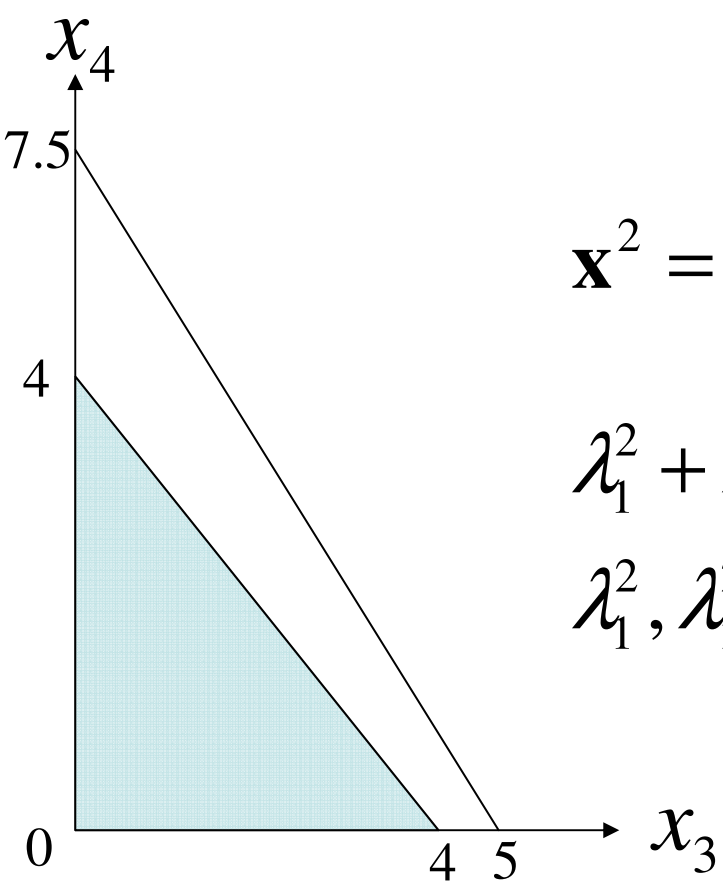
$$\mathbf{x}^1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda_1^1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \lambda_2^1 \begin{bmatrix} 0 \\ 10 \end{bmatrix} + \lambda_3^1 \begin{bmatrix} 2 \\ 6 \end{bmatrix} + \lambda_4^1 \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\lambda_1^1 + \lambda_2^1 + \lambda_3^1 + \lambda_4^1 = 1$$

$$\lambda_1^1, \lambda_2^1, \lambda_3^1, \lambda_4^1 \ge 0$$

Extreme points…

$$\mathbf{D}^2 = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \quad \mathbf{x}^2 = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} \quad \mathbf{d}^2 = \begin{bmatrix} 15 \\ 4 \end{bmatrix} \quad \Rightarrow \begin{array}{c} 3x_3 + 2x_4 \le 15 \\ x_3 + x_4 \le 4 \end{array}$$



$$\mathbf{x}^2 = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \lambda_1^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \lambda_2^2 \begin{bmatrix} 0 \\ 4 \end{bmatrix} + \lambda_3^2 \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$$

$$\lambda_1^2, \lambda_2^2, \lambda_3^2 \ge 0$$

Decomposition…

$$
\begin{aligned}
&z = \max \quad \mathbf{C}^1\mathbf{x}^1 + \mathbf{C}^2\mathbf{x}^2 \\
&s.t. \\
&\mathbf{D}^1\mathbf{x}^1 \le \mathbf{d}^1 \\
&\mathbf{D}^2\mathbf{x}^2 \le \mathbf{d}^2 \\
&\mathbf{A}^1\mathbf{x}^1 + \mathbf{A}^2\mathbf{x}^2 \le \mathbf{b}
\end{aligned}
$$

$$\equiv$$

$$
\begin{aligned}
&z = \max \quad \mathbf{C}^1\left(\sum_{t=1}^{|\mathbf{T}_1|} \lambda_t^1 \mathbf{x}_t^1\right) + \mathbf{C}^2\left(\sum_{t=1}^{|\mathbf{T}_2|} \lambda_t^2 \mathbf{x}_t^2\right) \\
&s.t. \\
&\sum_{t=1}^{|\mathbf{T}_1|} \lambda_t^1 = 1 \\
&\sum_{t=1}^{|\mathbf{T}_2|} \lambda_t^2 = 1 \\
&\mathbf{A}^1\left(\sum_{t=1}^{|\mathbf{T}_1|} \lambda_t^1 \mathbf{x}_t^1\right) + \mathbf{A}^2\left(\sum_{t=1}^{|\mathbf{T}_2|} \lambda_t^2 \mathbf{x}_t^2\right) \le \mathbf{b} \\
&\lambda_t^1 \ge 0, \quad t = 1, \ldots, |\mathbf{T}_1| \\
&\lambda_t^2 \ge 0, \quad t = 1, \ldots, |\mathbf{T}_2|
\end{aligned}
$$

Dantzig-Wolfe Reformulation

$$\mathbf{x}^1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2\lambda_3^1 + 4\lambda_4^1 \\ 10\lambda_2^1 + 6\lambda_3^1 \end{bmatrix}, \quad \lambda_1^1 + \lambda_2^1 + \lambda_3^1 + \lambda_4^1 = 1$$

$$\mathbf{x}^2 = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4\lambda_3^2 \\ 4\lambda_2^2 \end{bmatrix}, \quad \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$$

$$\mathbf{A}^1 = [8 \quad 6] \qquad \mathbf{A}^2 = [7 \quad 5] \qquad \mathbf{C}^1 = [90 \quad 80] \qquad \mathbf{C}^2 = [70 \quad 60] \qquad \mathbf{b} = [80]$$

$$z = \max \quad 90\left(2\lambda_3^1 + 4\lambda_4^1\right) + 80\left(10\lambda_3^1 + 6\lambda_4^1\right) + 70\left(4\lambda_3^2\right) + 60\left(4\lambda_2^2\right)$$

$s.t.$

$$\lambda_1^1 + \lambda_2^1 + \lambda_3^1 + \lambda_4^1 = 1$$

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$$

$$8\left(2\lambda_3^1 + 4\lambda_4^1\right) + 6\left(10\lambda_3^1 + 6\lambda_4^1\right) + 7\left(4\lambda_3^2\right) + 5\left(4\lambda_2^2\right) \le 80$$

$$\lambda_1^1, \lambda_2^1, \lambda_3^1, \lambda_4^1, \lambda_1^2, \lambda_2^2, \lambda_3^2 \ge 0$$

So, what’s the point here?

D-W reformulation looks more complicated!!

The point is…

$$\mathbf{x}^1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda_1^1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \lambda_2^1 \begin{bmatrix} 0 \\ 10 \end{bmatrix} + \lambda_3^1 \begin{bmatrix} 2 \\ 6 \end{bmatrix} + \lambda_4^1 \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\mathbf{x}^2 = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \lambda_1^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \lambda_2^2 \begin{bmatrix} 0 \\ 4 \end{bmatrix} + \lambda_3^2 \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

We can use column generation!!

Step 0: Construct Restricted LP with one extreme point each for x^1 and x^2.

Step 1: Solve Restricted LP.

Step 2: Get an extreme point for each x^1 and x^2 using their corresponding pricing subproblem.

Step 3: Append new extreme points to Restricted LP (if required), then go to Step 1.

Pricing subproblems…

Restricted Master LP

$$z = \max \quad \mathbf{C}^1\left(\sum_{t=1}^{|\hat{\mathbf{T}}_1|}\lambda_t^1\mathbf{x}_t^1\right) + \mathbf{C}^2\left(\sum_{t=1}^{|\hat{\mathbf{T}}_2|}\lambda_t^2\mathbf{x}_t^2\right)$$

$s.t.$

$$\sum_{t=1}^{|\hat{\mathbf{T}}_1|}\lambda_t^1 = 1 \quad (dual:\pi_1)$$

$$\sum_{t=1}^{|\hat{\mathbf{T}}_2|}\lambda_t^2 = 1 \quad (dual:\pi_2)$$

$$\mathbf{A}^1\left(\sum_{t=1}^{|\hat{\mathbf{T}}_1|}\lambda_t^1\mathbf{x}_t^1\right) + \mathbf{A}^2\left(\sum_{t=1}^{|\hat{\mathbf{T}}_2|}\lambda_t^2\mathbf{x}_t^2\right) \le \mathbf{b} \quad (dual:\rho)$$

$$\lambda_t^1 \ge 0, \quad t = 1,\ldots,|\hat{\mathbf{T}}_1|$$

$$\lambda_t^2 \ge 0, \quad t = 1,\ldots,|\hat{\mathbf{T}}_2|$$

Pricing subproblem for x^1

$$\max \quad z_1 = \mathbf{C}^1\mathbf{x}^1 - \rho\mathbf{A}^1\mathbf{x}^1 - \pi_1$$

$s.t.$

$$\mathbf{D}^1\mathbf{x}^1 \le \mathbf{d}^1$$

$$\mathbf{x}^1 \ge 0$$

Pricing subproblem for x^2

$$\max \quad z_2 = \mathbf{C}^2\mathbf{x}^2 - \rho\mathbf{A}^2\mathbf{x}^2 - \pi_2$$

$s.t.$

$$\mathbf{D}^2\mathbf{x}^2 \le \mathbf{d}^2$$

$$\mathbf{x}^2 \ge 0$$

Column generation

Start with $\hat{T}_1 = \{(0,0)\}, \quad \hat{T}_2 = \{(0,0)\}$

$$\max \quad z = 0$$

$$s.t.$$

$$\lambda_1^1 = 1 \quad (dual : \pi_1)$$

$$\lambda_1^2 = 1 \quad (dual : \pi_2)$$

$$0 \le 80 \quad (dual : \rho)$$

$$\lambda_1^1, \lambda_1^2 \ge 0$$

$$\text{Optimal solution} : (\rho, \pi_1, \pi_2) = (0,0,0)$$

Column generation

Pricing subproblem for x^1: $(\rho, \pi_1, \pi_2) = (0,0,0)$

$$\max \quad z_1 = (90x_1 + 80x_2) - \rho(8x_1 + 6x_2) - \pi_1$$

$$s.t.$$

$$3x_1 + x_2 \le 12$$

$$2x_1 + x_2 \le 10$$

$$x_1, x_2 \ge 0$$

An extreme point!!

$$\text{Optimal solution}: (x_1, x_2) = (0,10), \quad z_1 = 800$$

$$\text{Since } z_1 > 0, \text{ add } (0,10) \text{ to } \hat{T}_1.$$

Column generation

Pricing subproblem for x^2:

$$(\rho, \pi_1, \pi_2) = (0,0,0)$$

$$\max \quad z_2 = (70x_3 + 60x_4) - \rho(7x_3 + 5x_4) - \pi_2$$

$$s.t.$$

$$3x_3 + 2x_4 \le 15$$

$$x_3 + x_4 \le 4$$

$$x_3, x_4 \ge 0$$

An extreme point!!

$$\text{Optimal solution}: (x_3, x_4) = (4,0), \quad z_2 = 280$$

$$\text{Since } z_2 > 0, \text{ add } (4,0) \text{ to } \hat{T}_2.$$

Exercise:

Resolve the Restricted LP, and repeat the process…

Further reading…

Chapter 10.4